OPTIMAL STRESSES IN STRUCTURES

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In memory of Israel Gilad (1949 – 2005)

Abstract. For a given external loading on a structure we consider the optimal stresses. Ignoring the material properties the structure may have, we look for the distribution of internal forces or stresses that is in equilibrium with the external loading and whose maximal component is the least. We present an expression for this optimal value in terms of the external loading and the matrix relating the external degrees of freedom and the internal degrees of freedom. The implementation to finite element models consisting of elements of uniform stress distributions is presented. Finally, we give an example of stress optimization for of a two-element model of a cylinder under external traction.

1. INTRODUCTION

This paper presents an analysis of optimal stresses in structures under given loadings. In [1, 2, 3], optimal stress distributions for continuous bodies were considered. Although the problem for a continuous body is more difficult mathematically, the corresponding analysis for a structure having a finite number of degrees of freedom is more relevant for engineering applications.

From the point of view of statics, engineering structures—starting from simple trusses all the way to finite element models used for stress analysis of continuous bodies, and evidently, continuous models of bodies—are predominantly statically indeterminate. Mathematically, this means that we have more unknown parameters describing the stress distribution in the structure under consideration than equilibrium equations. This mathematical problem is solved usually by the introduction of constitutive relations and by coupling the statics problem with kinematics.

The work presented here takes a different approach to statically indeterminate problems. Remaining within the framework of statics, we do not specify any constitutive relations and look for the values of the unknown components that satisfy the equilibrium conditions and for which the maximal component is the least. Specifically, the problem may be stated as follows. Let f_d, $d = 1, \dots, D$, be the components of the known external loading vector on the structure where D is the number of degrees of freedom the structure

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has, and let φ_n, $n = 1, \ldots, N$, be the components of the unknown vector of internal forces—stress-like entities. As we consider statically indeterminate problems, $N > D$. The equations of equilibrium will be of the form

$$A^T(\varphi) = f, \quad \text{or} \quad A^T_{dn}\varphi_n = f_d, \tag{1.1}$$

where we use the summation convention and the reason we write the matrix as A^T rather than simply A will be made clear below. Thus, letting $\varphi_{\max} = \max_n |\varphi_n|$, we are looking for

$$S^{\text{opt}}_f = \min_{\varphi} \{\varphi_{\max}\}, \tag{1.2}$$

where the minimum is taken over all φ satisfying $A^T(\varphi) = f$.

Our basic result states that

$$S^{\text{opt}}_f = \max_w \frac{|f_d w_d|}{\sum_n |A_{nd} w_d|}, \tag{1.3}$$

where the maximum is taken over all global virtual displacement vectors $w = (w_1, \ldots, w_D)$ of the structure, so $f_d w_d$ is the virtual work performed by the external force vector.

A related quantity that we consider is the *stress sensitivity* of the structure defined as follows. Assuming that the internal forces have the same physical dimension as the external forces (dimensions of forces or forces divided by area), consider the ratio

$$K_f = \frac{S^{\text{opt}}_f}{\max_d |f_d|}. \tag{1.4}$$

Thus, K_f measures the sensitivity of the structure to the external force f. Next, we let the external force vary and we look for the worst possible ratio. The stress sensitivity of the structure is defined as

$$K = \max_f K_f = \max_f \left\{ \frac{S^{\text{opt}}_f}{\max_d |f_d|} \right\}. \tag{1.5}$$

It is shown in Section 3.5 that

$$K = \max_w \frac{\sum_d |w_d|}{\sum_n |A_{nd} w_d|}. \tag{1.6}$$

We emphasize that K is a geometric, kinematic property of the structure, i.e.., independent of material properties, loading conditions, etc.

The paper is outlined as follows. We start with the notation and basic facts regarding statically indeterminate structures. Then, we prove the results stated above. For the internal force vector φ, the value $\varphi_{\max}$ is represented as a norm, specifically, the dual of the norm $\|\chi\| = \sum_n |\chi_n|$ that we use for internal displacements χ. The basic tool we use is the norm preserving extension of functionals (the simplified, finite dimensional case of the Hahn-Banach theorem). Next, we present some details regarding the application of the method to the case of finite element models consisting of elements having uniform stress distributions. Finally, in a way of example, we consider the

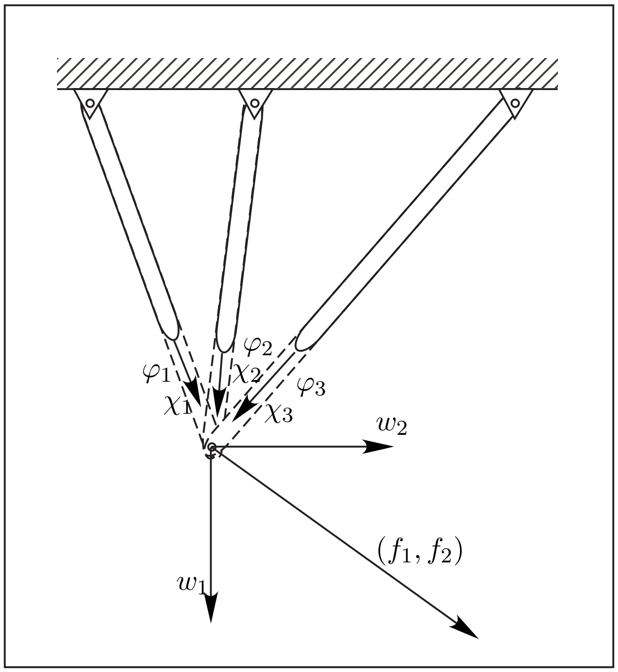

FIGURE 2.1. A SIMPLE INDETERMINATE STRUCTURE

case of a two elements model of a thick cylinder under external symmetric loading.

2. STATICALLY INDETERMINATE STRUCTURES

An elementary example for the type of structures we consider is shown in Fig. 1. Our method applies to a lot more complicated structures including a large variety of finite element models.

2.1. Kinematics. The structure is assumed to have D degrees of freedom. This means that we have a D-dimensional vector space $\mathcal{W}$ containing the *external* infinitesimal virtual displacements (generalized velocities) that are compatible with the displacements boundary conditions at the supports and the various constraints implied by the structural connections. A generic virtual displacement in $\mathcal{W}$ will be denoted by w. Thus in our model example of Fig. 1, the structure has two degrees of freedom and a generic virtual displacement is of the form $w = (w_1, w_2)$. Each external degree of freedom induces a base vector in $\mathcal{W}$.

Next we consider the space $\mathcal{S}$ containing noncompatible, or *internal*, infinitesimal virtual displacements of the structure. Such an internal deformation field of our model example is shown in Fig. 1. Here, the constraints of the structural interconnections of the various structural elements are not kept (the connection at the bottom joint in the figure). In particular, constant strains within structural elements may be represented as internal virtual displacement fields. A generic internal virtual displacement field will be denoted as χ. Clearly, internal deformations have more degrees of freedom than the external ones. We assume formally that N, the dimension of $\mathcal{S}$, is strictly larger than D. The various internal degrees of freedom induce base vectors in $\mathcal{S}$. (Returning to our model example, we note that the internal degrees of freedom, i.e, the base vectors in $\mathcal{S}$, may be unit changes in lengths of the bars or unit axial strains in the bars. It will be convenient also to define for a uniform strain structural member, such as a bar in the example, base vectors in $\mathcal{S}$ consisting of unit strain components in the element multiplied by its volume as in Section 4.)

An important role in the kinematics of the structure is played by the *interpolation mapping*

$$A\colon \mathcal{W} \longrightarrow \mathcal{S}. \tag{2.1}$$

This mapping associates an internal deformation vector with every external deformation. Clearly, as $N > D$, not all internal deformations may be obtained as images of external deformations under A. We further note that in case the structure is not supported distinct displacements fields that differ by a rigid displacement field induce the same strain field. While unsupported bodies may be considered following the methods of [2], we simplify the analysis and assume that the supports prevent such rigid displacement fields. Thus, we assume mathematically that the interpolation mapping is one-to-one, so the matrix of A is of full rank D.

2.2. **Statics.** An external force f performs virtual work (power) for various external displacements. Denoting by f_d the component of the force dual to the degree of freedom w_d, the virtual work may be written as

$$f(w) = f_d w_d. \tag{2.2}$$

In other words, we regard an external force as a linear functional

$$f\colon \mathcal{W} \longrightarrow \mathbb{R} \tag{2.3}$$

and the collection of all external forces is the dual space $\mathcal{W}^*$ containing all real valued linear mappings defined on $\mathcal{W}$.

In analogy, an internal force φ performs virtual work for virtual internal deformation fields. Denoting the component of the internal force φ corresponding to the component χ_n, by φ_n, $n = 1, \dots, N$, we write for the internal virtual work performed by φ for the virtual displacement χ

$$\varphi(\chi) = \varphi_n \chi_n. \tag{2.4}$$

Thus, an internal force is a linear mapping

$$\varphi\colon \mathcal{S} \longrightarrow \mathbb{R}, \tag{2.5}$$

i.e., φ belongs to the space $\mathcal{S}^*$ of real valued linear mapping on $\mathcal{S}$. Note that in case the component χ_n indicates a constant component of the strain in some structural element multiplied by its volume, then φ_n indicates the corresponding stress component (see Section 4).

The principle of virtual work serves as the condition for equilibrium within the framework of the structural model. Using the notation introduced above it states that

$$\varphi(A(w)) = f(w), \tag{2.6}$$

for all external vector fields w in $\mathcal{W}$. Using matrix notation where we keep the same symbol for a linear mapping (or a vector) and its corresponding matrix (or the corresponding column vector) the principle of virtual work is written as

$$\varphi^T A w = f^T w. \tag{2.7}$$

Thus, the principle of virtual work, or equivalently the equilibrium condition, may be written in any of the following forms

$$\varphi \circ A = f, \quad A^T \varphi = f, \quad A_{nd}\varphi_n = f_d, \quad A^*(\varphi) = f, \tag{2.8}$$

where in the last equation above we used the dual mapping $A^* \colon \mathcal{S}^* \longrightarrow \mathcal{W}^*$ defined by the condition $A^*(\varphi)(w) = \varphi(A(w))$ and whose matrix is the transpose of that of A as expected.

Given an external force f, the equilibrium conditions (2.8) provide a system of D equations for the N components of the internal force φ. As it was assumed that N is strictly larger than D and that A is one-to-one, this system of equations cannot determine φ uniquely. In fact, there is an $(N-D)$-dimensional vector space $\Phi = (A^*)^{-1}\{f\}$ of solutions to the equilibrium problem.

3. OPTIMAL SOLUTIONS

For given structures, where the material properties of the various structural elements are known, the constitutive relations provide the additional information so the internal force vector can be calculated uniquely for any given external force f. Here however, we consider the situation where no constitutive relations are given a-priori, and among all solutions φ in Φ, we look for the least bound on the maximal component.

Specifically, for each internal force φ, we set

$$\|\varphi\|^\infty = \varphi_{\max} = \max_n |\varphi_n|, \tag{3.1}$$

and we look for

$$S_f^{\text{opt}} = \min_\varphi \{\|\varphi\|^\infty\} = \min_\varphi \left\{\max_n |\varphi_n|\right\}, \tag{3.2}$$

where the minimum is taken over all internal forces φ satisfying $A^*(\varphi) = f$. Thus, for the case were the components of the internal force represent stresses in the various structural elements, we are looking for the least bound on the discretized approximating stress field.

In order to evaluate S_f^{opt} directly on the basis of its definition, one would have to generate the space of solutions Φ and then evaluate the optimal bound in that space. The analysis we present in the sequel will give an expression for S_f^{opt} that does not require the solution of the equilibrium equations (2.8).

3.1. **A Solution as an Extension of a Functional.** Since the interpolation mapping A is one-to-one, its inverse

$$A^{-1}\colon \operatorname{Image} A \longrightarrow \mathcal{W} \tag{3.3}$$

is well defined on its image, a subspace of $\mathcal{S}$. Given an external force f, we consider

$$\widehat{f} = f \circ A^{-1}\colon \operatorname{Image} A \longrightarrow \mathbb{R}, \tag{3.4}$$

a linear mapping defined on the subspace $\operatorname{Image} A$. Note that the equilibrium condition $f(w) = \varphi(A(w))$, for a linear functional φ on $\mathcal{S}$ may be written as

$$\varphi(\chi) = f \circ A^{-1}(\chi) = \widehat{f}(\chi) \tag{3.5}$$

for all compatible internal fields χ in $\operatorname{Image} A$. In other words, a solution of the equilibrium equations is a linear mapping φ defined on $\mathcal{S}$ that agrees with $\widehat{f} = f \circ A^{-1}$ on the subspace $\operatorname{Image} A$. Thus, φ is an *extension* of $\widehat{f}$ to the entire space $\mathcal{S}$.

It is noted that algebraically, given the linear functional $\widehat{f}$ on the subspace, it is straightforward to generate an extension of it to the entire space. In fact, it is sufficient to show this for the case where $N - D = 1$, so we have to extend $\widehat{f}$ to a space having one more dimension. In the general case where $N - D$ is any other finite number, the procedure can be carried out inductively adding one dimension at a time. To generate such an extension φ, one can choose an internal displacement vector χ_1 that does not belong to $\operatorname{Image} A$ and give an arbitrary value to $c = \varphi(\chi_1)$. Any vector χ in the larger space may be written as a linear combination

$$\chi = \chi_0 + a\chi_1 \tag{3.6}$$

for a compatible internal virtual displacement χ_0 in $\operatorname{Image} A$ and a real number a. Thus, for any extension φ we have

$$\begin{aligned} \varphi(\chi) &= \varphi(\chi_0 + a\chi_1) \\ &= \varphi(\chi_0) + a\varphi(\chi_1) \\ &= \widehat{f}(\chi_0) + ac. \end{aligned} \tag{3.7}$$

3.2. **Norms.** Recalling that we are looking for a solution of the equilibrium equations that minimizes $\|\varphi\|^{\infty} = \max_n |\varphi_n|$, we mention a number of useful properties of this norm on the space of internal forces. Consider the norm on the space $\mathcal{S}$ given by

$$\|\chi\|_1 = \sum_n |\chi_n|\,. \tag{3.8}$$

Then, the following holds

$$\|\varphi\|^{\infty} = \max_{\chi} \frac{|\varphi(\chi)|}{\|\chi\|_1}\,. \tag{3.9}$$

This relation between the norm for internal forces and the norm for internal virtual displacements is all we need. In fact, the following analysis applies to other criteria for optimization of the internal forces, i.e., criteria given by other norms say $\|\varphi\|$. To do this, one should determine the norm on the space of internal displacements to which $\|\varphi\|$ is dual, i.e., determine the norm $\|\chi\|$ such that

$$\|\varphi\| = \max_{\chi} \frac{|\varphi(\chi)|}{\|\chi\|}\,. \tag{3.10}$$

Thus, in the sequel we use a generic norm $\|\chi\|$ for internal displacements and the corresponding dual norm $\|\varphi\|$ for internal forces satisfying Equation (3.10). In fact, once the norm $\|\chi\|$ is established as the one for which our optimality condition is a dual norm, the norm on the space of internal displacement fields (rather than the one on the space of internal forces) plays the central role as will be seen below. It is noted that if the optimality criterion is given in terms of a norm $\|\varphi\|$, the associated norm on the space of internal displacements can be found by (see [4, p. 186])

$$\|\chi\| = \max_{\varphi} \frac{|\varphi(\chi)|}{\|\varphi\|}\,. \tag{3.11}$$

For external virtual displacements we may also consider the norm

$$\|w\|_1 = \sum_d |w_d|\,, \tag{3.12}$$

and use the dual norm

$$\|f\|^{\infty} = \max_{w} \frac{|f(w)|}{\|w\|_1} = \max_{d} |f_d| \tag{3.13}$$

for external force vectors.

Contrary to the previous paragraph, we will find it useful in the analysis of optimal internal forces below to use for external virtual displacements the norm

$$\|w\| = \|A(w)\|, \tag{3.14}$$

where on the right we use the norm on $\mathcal{S}$. The corresponding dual norm for external forces is therefore

$$\|f\| = \max_{w} \frac{|f(w)|}{\|w\|} = \max_{w} \frac{|f(w)|}{\|A(w)\|}\,. \tag{3.15}$$

3.3. **Optimal Extensions.** Considering the linear functional $\widehat{f} = f \circ A^{-1} \colon \text{Image}\, A \to \mathbb{R}$, we may evaluate its dual norm relative to the norm (3.8) on Image A. Thus,

$$\|\widehat{f}\|^{\infty} = \max_{\chi \in \text{Image}\, A} \frac{\left|\widehat{f}(\chi)\right|}{\|\chi\|_1} \tag{3.16}$$

where it is noted that the maximum is evaluated for all χ in Image A (and not the entire space $\mathcal{S}$). The Hahn-Banach theorem of functional analysis states that there is a linear functional $\varphi_f^{\text{opt}} \colon \mathcal{S} \longrightarrow \mathbb{R}$ (i.e., defined on the space $\mathcal{S}$) such that

$$\varphi_f^{\text{opt}}(\chi) = \widehat{f}(\chi) \tag{3.17}$$

for all χ in Image A, and

$$\|\varphi_f^{\text{opt}}\|^{\infty} = \max_{\chi \in \mathcal{S}} \frac{\left|\varphi_f^{\text{opt}}(\chi)\right|}{\|\chi\|_1} = \|\widehat{f}\|^{\infty} = \max_{\chi \in \text{Image}\, A} \frac{\left|\widehat{f}(\chi)\right|}{\|\chi\|_1}. \tag{3.18}$$

In other words, φ extends $\widehat{f}$ without increasing its norm. It was mentioned earlier that extending the functional one dimension at a time is simple (see Equation (3.7). Appendix A presents the construction for the addition of one dimension to the domain without increasing the norm of the linear functional. In fact, the Hahn-Banach theorem asserts that this can be done for infinite dimensional spaces also and is used in the continuum counterpart of this analysis in [1, 2, 3].

We end this subsection by noting that

$$\|\widehat{f}\|^{\infty} = \max_{\chi \in \text{Image}\, A} \frac{\left|\widehat{f}(\chi)\right|}{\|\chi\|_1} = \max_{w \in \mathcal{W}} \frac{|f(w)|}{\|A(w)\|} = \|f\|. \tag{3.19}$$

3.4. **The Equation for the Optimum.** Returning to the expression for the optimum, we note that in general for any internal force φ that extends $\widehat{f}$

$$\|\widehat{f}\|^{\infty} = \max_{\chi \in \text{Image}\, A} \frac{\left|\widehat{f}(\chi)\right|}{\|\chi\|} \tag{3.20}$$

$$= \max_{\chi \in \text{Image}\, A} \frac{|\varphi(\chi)|}{\|\chi\|} \quad \text{(by the principle of virtual work)} \tag{3.21}$$

$$\leqslant \max_{\chi \in \mathcal{S}} \frac{|\varphi(\chi)|}{\|\chi\|} = \|\varphi\|^{\infty}. \tag{3.22}$$

Thus, since for φ_f^{opt}, $\|\varphi_f^{\text{opt}}\|^{\infty} = \|\widehat{f}\|^{\infty} = \|f\|$, as in Eqs. (3.18) and (3.19), we have

$$\|\varphi_f^{\text{opt}}\|^{\infty} = \|f\| = \min_{\varphi} \|\varphi\|^{\infty}, \tag{3.23}$$

where the minimum is taken over all extensions φ of $\widehat{f}$. Since an extension φ of $\widehat{f}$ satisfies the equilibrium condition as in Eq. (3.5), we conclude that

$$\min_{f=A^*(\varphi)} \left\{ \max_n \left| (\varphi_f^{\text{opt}})_n \right| \right\} = \|\varphi_f^{\text{opt}}\|^\infty = \|f\|. \tag{3.24}$$

It follows that

$$S_f^{\text{opt}} = \max_{w\in\mathcal{W}} \frac{|f(w)|}{\|A(w)\|} = \max_{w\in\mathcal{W}} \frac{|f_d w_d|}{\sum_n |A_{nd} w_d|}\,. \tag{3.25}$$

3.5. **The Stress Sensitivity.** We now derive the expression for the stress sensitivity of the structure. Recalling its definition in Eq. (1.5)

$$K = \max_f \left\{ \frac{S_f^{\text{opt}}}{\max_d |f_d|} \right\},$$

we substitute the expression for the optimum internal force to get

$$K = \max_f \left\{ \frac{1}{\max_d |f_d|} \left\{ \max_{w\in\mathcal{W}} \frac{|f_d w_d|}{\sum_n |A_{nd} w_d|} \right\} \right\} \tag{3.26}$$

$$= \max_{w\in\mathcal{W}} \left\{ \frac{1}{\sum_n |A_{nd} w_d|} \left\{ \max_f \frac{|f_d w_d|}{\max_d |f_d|} \right\} \right\}. \tag{3.27}$$

We may use now Eq. (3.11) and arrive at

$$K = \max_{w\in\mathcal{W}} \frac{\sum_d |w_d|}{\sum_n |A_{nd} w_d|}\,. \tag{3.28}$$

4. APPLICATION TO FINITE ELEMENTS

A typical situation where one would like to apply the foregoing analysis is a finite element model of a continuous body. Thus, we briefly describe here some additional details for the simple situation of a finite element model where it is assumed that the stress is uniform within each element. We do not consider here the question of approximation of the solution to the continuum problem by finite elements and take the finite element model as a given structure.

Let L be the number of elements, e_l the l-th element, and σ_{ij}^l the components of the uniform stress in that element. We want the collection of σ_{ij}^l for the various elements l and various components i, j to be the components of our internal force vector. Thus, the index n is replaced by the collection of 3 indices l, i, j, $\varphi_{ij}^l = \sigma_{ij}^l$, and

$$\|\varphi\| = \max_{i,j,l} \left| \sigma_{ij}^l \right| \tag{4.1}$$

is the quantity that we want to minimize.

The internal degrees of freedom χ_{ij}^l should be chosen such that $\sum_{l,i,j} \sigma_{ij}^l \chi_{ij}^l$ is the virtual work of the internal force for the given internal displacements.

Writing

$$U = \int_{\cup_l e_l} \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = \sum_l \sigma^l_{ij} \int_{e_l} \varepsilon_{ij}\,\mathrm{d}V, \tag{4.2}$$

where ε_{ij} denotes the linear strain field, one realizes that the internal degrees of freedom are given by

$$\chi^l_{ij} = \int_{e_l} \varepsilon_{ij}\,\mathrm{d}V. \tag{4.3}$$

For uniform strain elements $\chi^l_{ij} = V_l \varepsilon^l_{ij}$ (no sum on l), where ε^l_{ij} are the components of the uniform strain in the element e_l and V_l is its volume. The norm $\|\chi\|$ is therefore

$$\|\chi\| = \sum_{i,j,l} \left| \int_{e_l} \varepsilon_{ij}\,\mathrm{d}V \right| \tag{4.4}$$

and for uniform strain elements

$$\|\chi\| = \sum_{i,j,l} \left|\varepsilon^l_{ij}\right| V_l. \tag{4.5}$$

It is a standard procedure in the construction of finite element models to use an array E^l_{ijd} such that

$$\chi^l_{ij} = \int_{e_l} \varepsilon_{ij}\,\mathrm{d}V = E^l_{ijd} w_d, \tag{4.6}$$

where the array E^l_{ijd} is generated using the relation between the external degrees of freedom and the displacements at the nodes of the the element e_l, using the shape functions to obtain the displacement field within the element, and using the discrete forms of the differentiation and integration operators in order to obtain the strain components and their integrals. Thus, the array E^l_{ijd} replaces the matrix A_{nd} in our expression for the optimum. We conclude that

$$S^{\mathrm{opt}}_f = \|\varphi^{\mathrm{opt}}_f\| = \min_\sigma \left\{ \max_{l,i,j} \left|\sigma^l_{ij}\right| \right\} = \max_{w\in\mathcal{W}} \frac{|f_d w_d|}{\sum_{l,i,j}\left|E^l_{ijd} w_d\right|}\,. \tag{4.7}$$

5. EXAMPLE

We consider the following 2-dimensional example. The structure is a finite element model of a cylinder of inner radius r_1 and outer radius r_3 under external normal traction p at the outer boundary (see Fig. 2).

The finite element model consists of two uniform stress elements e_1 and e_2, corresponding to the regions $r_1 \leqslant r \leqslant r_2$, and $r_2 \leqslant r \leqslant r_3$, respectively, where $r_2 = (r_1 + r_3)/2$. Due to the cylindrical symmetry the problem has three degrees of freedom and a typical external virtual displacement is of

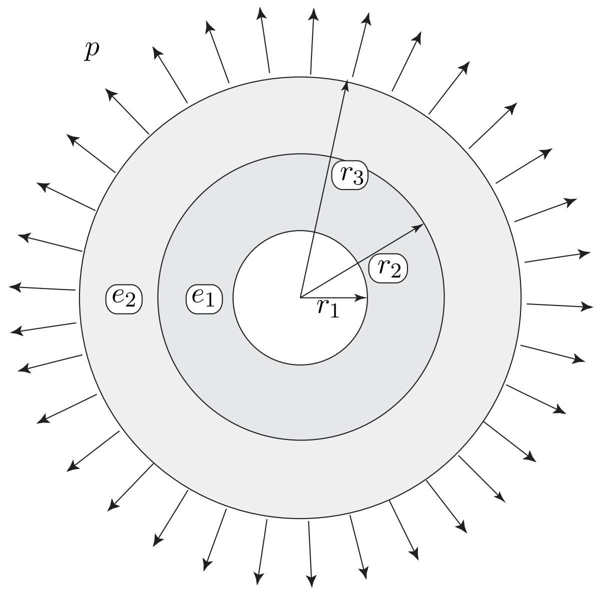

FIGURE 5.1. A 2 ELEMENT MODEL OF A CYLINDER

the form $w = (w_1, w_2, w_3)$, where w_d is the radial displacement at r_d. The corresponding components of an external force f are given as

$$f_d = 2\pi r_d p_d, \quad \text{(no summation)}, \quad d = 1, 2, 3, \tag{5.1}$$

where, p_d is the applied load at r_d. Thus, for the case under consideration $f = (0, 0, 2\pi r_3 p)$. The space of internal virtual displacements $\mathcal{S}$ will be 4-dimensional and an internal displacement will be of the form $\chi_1 = S_1(\varepsilon_r)_1$, $\chi_2 = S_1(\varepsilon_\theta)_1$, $\chi_3 = S_2(\varepsilon_r)_2$, $\chi_4 = S_2(\varepsilon_\theta)_2$, where S_l is the area of the l-th element and $(\varepsilon_r)_l$, $(\varepsilon_\theta)_l$ are the uniform strain components in e_l. The values of the strain components within the elements are approximated as

$$(\varepsilon_r)_l = \frac{w_{l+1} - w_l}{r_{l+1} - r_l}, \quad (\varepsilon_\theta)_l = \frac{w_l + w_{l+1}}{2m_l}, \tag{5.2}$$

where $m_l = (r_l + r_{l+1})/2$ denotes the mean radius of the l-th element. The matrix A is easily calculated to give

$$[A] = \begin{bmatrix} -\frac{S_1}{r_2 - r_1} & \frac{S_1}{r_2 - r_1} & 0 \\ \frac{S_1}{2m_1} & \frac{S_1}{2m_1} & 0 \\ 0 & -\frac{S_2}{r_3 - r_2} & \frac{S_2}{r_3 - r_2} \\ 0 & \frac{S_2}{2m_2} & \frac{S_2}{2m_2} \end{bmatrix}. \tag{5.3}$$

Instead of the optimal stress, it will be convenient to determine

$$\frac{1}{S^{\text{opt}}} = \min_{w} \frac{\sum_n |A_{nd} w_d|}{|f_d w_d|}\,. \tag{5.4}$$

Since both numerator and denominator are homogeneous in the vector w, we normalize it by conducting the search for the minimum over all displacement vectors satisfying

$$f_d w_d = f_3 w_3 = 1 \tag{5.5}$$

so

$$w_3 = \frac{1}{2\pi r_3 p}\,, \tag{5.6}$$

and we have to minimize

$$\sum_n |A_{nd} w_d| \tag{5.7}$$

over all values of (w_1, w_2) subject to the condition (5.6). Writing the sum above explicitly and using the fact that $A_{31} = A_{32} = 0$, we have

$$\frac{1}{S^{\text{opt}}} = \min_{w_2} \left\{ P + |A_{3d} w_d| + |A_{4d} w_d| \right\}, \tag{5.8}$$

where

$$P = \min_{w_1} \left\{ |A_{11} w_1 + A_{12} w_2| + |A_{21} w_1 + A_{22} w_2| \right\}. \tag{5.9}$$

As a function of w_1, the expression in the curly brackets above is piecewise affine and attains its minimum at some point where two adjacent line segments of its graph meet—at some value of w_1 where one of the absolute value terms vanishes. This gives

$$P = \frac{|w_2|\, |A_{11} A_{22} - A_{12} A_{21}|}{\max\left\{|A_{11}|\,, |A_{21}|\right\}} = \frac{|w_2|\, |A_{11} A_{22} - A_{12} A_{21}|}{|A_{11}|}\,, \tag{5.10}$$

as evidently $|A_{11}| > |A_{21}|$. Substituting the values of the various components of the matrix we obtain

$$P = 2\pi(r_2 - r_1)\, |w_2| \tag{5.11}$$

We can turn back to the minimization of Eq. (5.8) where now the minimum is attained at a value of w_2 where one of the two absolute value terms vanishes or at the value where P vanishes (so $w_2 = 0$). Setting

$$Q = P + |A_{3d} w_d| + |A_{4d} w_d|\,, \tag{5.12}$$

for the case $A_{32} w_2 + A_{33} w_3 = 0$, we have

$$Q = Q_1 = \frac{|w_3|}{|A_{32}|} \left\{ |A_{33}| \frac{D_1}{\max\left\{|A_{11}|\,, |A_{21}|\right\}} + D_2 \right\}, \tag{5.13}$$

where $D_1 = A_{11} A_{22} - A_{12} A_{21}$, and $D_2 = A_{33} A_{42} - A_{43} A_{32} = 2\pi S_2$ are the determinants of the two submatrices. It follows that

$$Q_1 = 2\pi(r_3 - r_1)\, |w_3|\,. \tag{5.14}$$

For the case $w_2 = 0$, we have

$$Q = Q_2 = (|A_{33}| + |A_{43}|)\, |w_3| = 2\pi r_3\, |w_3|\,, \tag{5.15}$$

and for the case $A_{42}w_2 + A_{43}w_3 = 0$, we have

$$\begin{aligned} Q = Q_3 &= |w_3| \left\{ \frac{|A_{43}|}{|A_{42}|} \frac{D_1}{\max\{|A_{11}|, |A_{21}|\}} + \frac{D_2}{|A_{42}|} \right\} \\ &= 2\pi(r_3 + 2r_2 - r_1)\, |w_3| \,. \end{aligned} \tag{5.16}$$

It follows that

$$\frac{1}{S^{\text{opt}}} = \min_{w_2} \{Q_1, Q_2, Q_3\} = Q_1 \tag{5.17}$$

and substituting the normalization condition on w_3, we finally obtain

$$S^{\text{opt}} = \frac{r_3}{r_3 - r_1}\, p.$$

It is noted that this value corresponds to a uniform value of σ_θ that will balance the external loading for one half of the cylinder.

This result for a common engineering problem could have been possibly obtained by direct analysis of the hollow cylinder. However, this example was aimed at demonstrating the strength of the proposed procedure and the ability to implement it in real life structures. Clearly, aside from the practical significance of knowing the optimal stress that may develop in a structure under given loading conditions, this procedure can be used to assess the optimality of a given standard design. Thus, comparing of the maximal stresses developing in a proposed design with the optimal value obtained by application of the formulation outlined above, one can estimate how much the design can be improved.

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Appendix A. Appendix: Norm Preserving Extensions

Let $\mathcal{W}_0 \subset \mathcal{W}$ be a vector subspace and $F_0 \colon \mathcal{W}_0 \longrightarrow \mathbb{R}$ a linear functional such that

$$F_0(w_0) \leqslant \|F_0\| \|w_0\|, \quad \text{for all} \quad w_0 \in \mathcal{W}_0. \tag{A.1}$$

Let w_1 be an element of $\mathcal{W} - \mathcal{W}_0$ and let $\mathcal{W}_1$ be the vector space spanned by $\mathcal{W}_0$ and w_1, i.e.,

$$\mathcal{W}_1 = \{w_0 + aw_1;\, a \in \mathbb{R},\, w_0 \in \mathcal{W}_0\}\,.$$

We want to extend F_0 to a functional

$$F_1 \colon \mathcal{W}_1 \longrightarrow \mathbb{R},$$

such that

$$F_1(w_0) = F_0(w_0), \quad \text{for all} \quad w_0 \in \mathcal{W}_0 \tag{A.2}$$

$$F_1(w) \leqslant \|F_0\|\|w\|, \quad \text{for all} \quad w \in \mathcal{W}_1. \tag{A.3}$$

Assuming F_1 satisfies the above requirements, then,

$$F_1(w_0 + w_1) = F_1(w_0) + F_1(w_1) \tag{A.4}$$

$$\|F_0\|\|w_0 + w_1\| \geqslant F_0(w_0) + F_1(w_1) \qquad \text{using } \|F_1\| = \|F_0\|. \tag{A.5}$$

Thus,

$$F_1(w_1) \leqslant \|F_0\|\|w_0 + w_1\| - F_0(w_0) \quad \text{for all} \quad w_0 \in \mathcal{W}_0. \tag{A.6}$$

Similarly,

$$F_1(-w_0 - w_1) = -F_1(w_0) - F_1(w_1) \tag{A.7}$$

$$\|F_0\|\| - w_0 - w_1\| \geqslant -F_0(w_0) - F_1(w_1) \tag{A.8}$$

$$\|F_0\|\|w_0 + w_1\| \geqslant -F_0(w_0) - F_1(w_1). \tag{A.9}$$

Thus,

$$F_1(w_1) \geqslant -\|F_0\|\|w_0 + w_1\| - F_0(w_0) \quad \text{for all} \quad w_0 \in \mathcal{W}_0. \tag{A.10}$$

We now show that the necessary conditions of Eqs. (A.6) and (A.10) are also sufficient. For any $w = w_0 + aw_1 \in \mathcal{W}_1$ we have

$$F_1(w_0 + aw_1) = aF_0(w_0/a) + aF_1(w_1), \tag{A.11}$$

using $F_1(w_0) = F_0(w_0)$.

Now if $a > 0$, we may use Eq. (A.6) for w_0/a in the second term on the right of Eq. (A.11) to get

$$\begin{aligned} F_1(w_0 + aw_1) &\leqslant aF_0(w_0/a) + a\left(\|F_0\|\|w_0/a + w_1\| - F_0(w_0/a)\right) \\ &= \|F_0\|\|w_0 + aw_1\|. \end{aligned}$$

Alternatively, for $a < 0$, Equation (A.10) may be rewritten as

$$aF_1(w_1) \leqslant -a\|F_0\|\|w_0 + w_1\| - aF_0(w_0) \quad \text{for all} \quad w_0 \in \mathcal{W}_0.$$

When we substitute this for w_0/a in Equation (A.11) we obtain

$$\begin{aligned} F_1(w_0 + aw_1) &\leqslant aF_0(w_0/a) - a\|F_0\|\|w_0/a + w_1\| - aF_0(w_0/a), \\ &= -a\|F_0\|\|w_0/a + w_1\|, \\ &= \|F_0\|\|w_0 + aw_1\|, \end{aligned}$$

where we used the fact that $(-a) > 0$ in the last line above. This completes the proof.

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