A CHARACTERIZATION OF X FOR WHICH SPACES $C_p(X)$ ARE DISTINGUISHED AND ITS APPLICATIONS

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Abstract. We prove that the locally convex space $C_p(X)$ of continuous real-valued functions on a Tychonoff space X equipped with the topology of pointwise convergence is distinguished if and only if X is a Δ-space in the sense of [21]. As an application of this characterization theorem we obtain the following results:

1) If X is a Čech-complete (in particular, compact) space such that $C_p(X)$ is distinguished, then X is scattered. 2) For every separable compact space of the Isbell–Mrówka type X, the space $C_p(X)$ is distinguished. 3) If X is the compact space of ordinals $[0, \omega_1]$, then $C_p(X)$ is not distinguished.

We observe that the existence of an uncountable separable metrizable space X such that $C_p(X)$ is distinguished, is independent of ZFC. We explore also the question to which extent the class of Δ-spaces is invariant under basic topological operations.

1. Introduction

Following J. Dieudonné and L. Schwartz [8] a locally convex space (lcs) E is called *distinguished* if every bounded subset of the bidual of E in the weak*-topology is contained in the closure of the weak*-topology of some bounded subset of E. Equivalently, a lcs E is distinguished if and only if the strong dual of E (i.e. the topological dual of E endowed with the strong topology) is *barrelled*, (see [19, 8.7.1]). A. Grothendieck [17] proved that a metrizable lcs E is distinguished if and only if its strong dual is *bornological*. We refer the reader to survey articles [5] and [6] which present several more modern results about distinguished metrizable and Fréchet lcs.

Throughout the article, all topological spaces are assumed to be Tychonoff and infinite. By $C_p(X)$ and $C_k(X)$ we mean the spaces of all real-valued continuous functions on a Tychonoff space X endowed with the topology of pointwise convergence and the compact-open topology, respectively. By a *bounded set* in a topological vector space (in particular, $C_p(X)$) we understand any set which is absorbed by every 0-neighbourhood.

For spaces $C_p(X)$ we proved in [13] the following theorem (the equivalence (1) $\Leftrightarrow$ (4) has been obtained in [11]).

Theorem 1.1. *For a Tychonoff space X, the following conditions are equivalent*:

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(1) *$C_p(X)$ is distinguished.*
(2) *$C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e. for every bounded set A in $\mathbb{R}^X$ there exists a bounded set B in $C_p(X)$ such that $A \subset cl_{\mathbb{R}^X}(B)$.*
(3) *For every $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$.*
(4) *The strong dual of the space $C_p(X)$ carries the finest locally convex topology.*

Several examples of $C_p(X)$ with(out) distinguished property have been provided in papers [11], [12] and [13]. The aim of this research is to continue our initial work on distinguished spaces $C_p(X)$.

The following concept plays a key role in our paper. We show its applicability for the studying of distinguished spaces $C_p(X)$.

Definition 1.2. ([21]) *A topological space X is said to be a Δ-space if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \omega$.*

We should mention that R. W. Knight [21] called all topological spaces X satisfying the above Definition 1.2 by Δ-sets. The original definition of a Δ-set of the real line $\mathbb{R}$ is due to G. M. Reed and E. K. van Douwen (see [27]). In this paper, for general topological spaces satisfying Definition 1.2 we reserve the term Δ-*space*. The class of all Δ-spaces is denoted by Δ.

In Section 2 we give an intrinsic description of all spaces $X \in \Delta$. One of the main results of our paper, Theorem 2.1 says that X is a Δ-space if and only if $C_p(X)$ is a distinguished space. This characterization theorem has been applied systematically for obtaining a range of results from our paper.

Our main result in Section 3 states that a Čech-complete (in particular, compact) $X \in \Delta$ must be scattered. A very natural question arises what are those scattered compact spaces $X \in \Delta$. In view of Theorem 2.1, it is known that a Corson compact X belongs to the class Δ if and only if X is a scattered Eberlein compact space [13]. With the help of Theorems 2.1 and 3.8 we show that the class Δ contains also all separable compact spaces of the Isbell–Mrówka type. Nevertheless, as we demonstrate in Section 3, there are compact scattered spaces $X \notin \Delta$ (for example, the compact space $[0, \omega_1]$).

Section 4 deals with the questions about metrizable spaces $X \in \Delta$. We notice that every σ-scattered metrizable space X belongs to the class Δ. For separable metrizable spaces X, our analysis reveals a tight connection between distinguished $C_p(X)$ and well-known set-theoretic problems about special subsets of the real line $\mathbb{R}$. We observe that the existence of an uncountable separable metrizable space $X \in \Delta$ is independent of ZFC and it is equivalent to the existence of a separable countably paracompact nonnormal Moore space. We refer readers to [24] for the history of the normal Moore problem.

In Section 5 we study whether the class Δ is invariant under the basic topological operations: subspaces, (quotient) continuous images, finite/countable unions and finite products. We pose several new open problems.

2. Description Theorem

In this section we provide an intrinsic description of $X \in \Delta$. For the reader's convenience we recall some relevant terminology.

(a) A disjoint cover $\{X_\gamma : \gamma \in \Gamma\}$ of X is called a *partition* of X.

(b) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an *expansion* of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in X if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.
(c) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called *point-finite* if no point belongs to infinitely many U_γ -s.

Theorem 2.1. *For a Tychonoff space X, the following conditions are equivalent*:
(1) *$C_p(X)$ is distinguished.*
(2) *Any countable partition of X admits a point-finite open expansion in X.*
(3) *Any countable disjoint collection of subsets of X admits a point-finite open expansion in X.*
(4) *X is a Δ-space.*

Proof. Observe that every collection of pairwise disjoint subsets of X, $\{X_\gamma : \gamma \in \Gamma\}$ can be extended to a partition by adding a single set $X_* = X \setminus \bigcup\{X_\gamma : \gamma \in \Gamma\}$. If the obtained partition admits a point-finite open expansion in X, then removing one open set we get a point-finite open expansion of the original disjoint collection. This shows evidently the equivalence (2) $\Leftrightarrow$ (3).

Assume now that (3) holds. Let $\{D_n : n \in \omega\}$ be a decreasing sequence subsets of X with empty intersection. Define $X_n = D_n \setminus D_{n+1}$ for each $n \in \omega$. By assumption, a disjoint collection $\{X_n : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in X. Then $\{V_n = \bigcup\{U_i : i \geq n\} : n \in \omega\}$ is an open decreasing expansion in X with empty intersection. This proves the implication (3) $\Rightarrow$ (4).

Next we show (4) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of X. Define $D_0 = X$ and $D_n = X \setminus \bigcup\{X_i : i < n\}$. Then $X_n \subset D_n$ for every n, the sequence $\{D_n : n \in \omega\}$ is decreasing and its intersection is empty. Assuming (4), we find an open decreasing expansion $\{U_n : n \in \omega\}$ of $\{D_n : n \in \omega\}$ in X such that $\bigcap\{U_n : n \in \omega\} = \emptyset$. For every $x \in X$ there is n such that $x \notin U_m$ for each $m > n$, it means that $\{U_n : n \in \omega\}$ is a point-finite expansion of $\{X_n : n \in \omega\}$ in X. This finishes the proof (3) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Now we prove the implication (1) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of X. Fix any function $f \in \mathbb{R}^X$ which satisfies the following conditions: for each $n \in \omega$ and every $x \in X_n$ the value of $f(x)$ is greater than n. By assumption, there is a bounded subset B of $C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$. Hence, for every $n \in \omega$ and every point $x \in X_n$, there exists $f_x \in B$ such that $f_x(x) > n$. But f_x is a continuous function, therefore there is an open neighbourhood $U_x \subset X$ of x such that $f_x(y) > n$ for every $y \in U_x$. We define an open set $U_n \subset X$ as follows: $U_n = \bigcup\{U_x : x \in X_n\}$. Evidently, $X_n \subseteq U_n$ for each $n \in \omega$. If we assume that the open expansion $\{U_n : n \in \omega\}$ is not point-finite, then there exists a point $y \in X$ such that there are infinitely many numbers n with $y \in U_{x_n}$ for some $x_n \in X_n$. This means that $\sup\{g(y) : g \in B\} = \infty$, which contradicts the boundedness of B.

It remains to prove (2) $\Rightarrow$ (1). By Theorem 1.1, we need to show that for every mapping $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$. If there exists a constant $r > 0$ such that $\sup\{|f(x)| : x \in X\} < r$, then we take $B = \{h \in C(X) : \sup\{|h(x)| : x \in X\} < r\}$. It is easy to see that B is as required.

Let $f \in \mathbb{R}^X$ be unbounded. Denote by $Y_0 = \emptyset$ and $Y_n = \{x \in X : n-1 \leq |f(x)| < n\}$ for each non-zero $n \in \omega$. Define $\varphi : X \to \omega$ by the rule: if $Y_n \neq \emptyset$ then $\varphi(x) = n$ for every $x \in Y_n$. So, $|f| < \varphi$. Put $X_n = \varphi^{-1}(n)$ for each $n \in \omega$. Note that some sets X_n might happen to be empty, but the collection $\{X_n : n \in \omega\}$ is a partition of X with countably many nonempty X_n -s. By our assumption, there exists a point-finite open expansion $\{U_n : n \in \omega\}$ of the partition $\{X_n : n \in \omega\}$.

Define $F : X \to \omega$ by $F(x) = \max\{n : x \in U_n\}$. Obviously, $f < F$. Finally, we define $B = \{h \in C_p(X) : |h| \leq F\}$. Then $f \in cl_{\mathbb{R}^X}(B)$, because for every finite subset $K \subset X$ there is a function $h \in B$ such that $f \restriction_K = h \restriction_K$. Indeed, given a finite subset $K \subset X$, let $\{V_x : x \in K\}$ be the family of pairwise disjoint open sets such that $x \in V_x \subset U_{\varphi(x)}$ for every $x \in K$. For each $x \in K$, fix a continuous function $h_x : X \to [-\varphi(x), \varphi(x)]$ such that $h_x(x) = f(x)$ and h_x is equal to the constant value 0 on the closed set $X \setminus V_x$. One can verify that $h = \Sigma_{x \in K} h_x \in B$ is as required. □

Below we present a straightforward application of Theorem 2.1.

Corollary 2.2. ([13]) *Let Z be any subspace of X. If X belongs to the class Δ, then Z also belongs to the class Δ.*

Proof. If $\{Z_\gamma : \gamma \in \Gamma\}$ is any collection of pairwise disjoint subsets of Z and there exists a point-finite open expansion $\{U_\gamma : \gamma \in \Gamma\}$ in X, then obviously $\{U_\gamma \cap Z : \gamma \in \Gamma\}$ is a point-finite expansion consisting of the sets relatively open in Z. It remains to apply Theorem 2.1. □

The last result can be reversed, assuming that $X \setminus Z$ is finite.

Proposition 2.3. *Let Z be a subspace of X such that $Y = X \setminus Z$ is finite. If Z belongs to the class Δ, then X belongs to Δ as well.*

Proof. Let $\{X_n : n \in \omega\}$ be any countable collection of pairwise disjoint subsets of X. Denote by F the set of those $n \in \omega$ such that $X_n \bigcap Y \neq \emptyset$. There might be only finitely many X_n -s which intersect the finite set Y, hence $F \subset \omega$ is finite. If $n \in F$, then we simply declare that U_n is equal to X. Consider the subcollection $\{X_n : n \in \omega \setminus F\}$. It is a countable collection of pairwise disjoint subsets of Z. Since $Z \in \Delta$, by Theorem 2.1, there is a point-finite open expansion $\{U_n : n \in \omega \setminus F\}$ in Z. Observe that Z is open in X, therefore all those U_n -s remain open in X. Bringing all U_n -s of both sorts together we obtain a point-finite open expansion $\{U_n : n \in \omega\}$ in X. Finally, $X \in \Delta$, by Theorem 2.1. □

Remark 2.4. The following applicable concept has been re-introduced in [13]. A family $\{\mathcal{N}_x : x \in X\}$ of subsets of a Tychonoff space X is called a *scant cover* for X if each $\mathcal{N}_x$ is an open neighbourhood of x and for each $u \in X$ the set $X_u = \{x \in X : u \in \mathcal{N}_x\}$ is finite. [1]

Our Theorem 2.1 generalizes one of the results obtained in [13] stating that if X admits a scant cover $\{\mathcal{N}_x : x \in X\}$ then $C_p(X)$ is distinguished. Indeed, let $\{X_\gamma : \gamma \in \Gamma\}$ be any collection of pairwise disjoint subsets of X. Define $U_\gamma = \bigcup\{\mathcal{N}_x : x \in X_\gamma\}$. It is easily seen that $\{U_\gamma : \gamma \in \Gamma\}$ is a point-finite open expansion in X, by definition of a scant cover. Applying Theorem 2.1, we conclude that $C_p(X)$ is distinguished.

3. Applications to compact spaces $X \in \Delta$

First we recall a few definitions and facts (probably well-known) which will be used in the sequel. A space X is said to be *scattered* if every nonempty subset A of X has an isolated point in A. Denote by $A^{(1)}$ the set of all non-isolated (in A)

[1] The referee kindly informed the authors that this notion also is known in the literature under the name *the point-finite neighbourhood assignment*.

points of $A \subset X$. For ordinal numbers α, the α-th derivative of a topological space X is defined by transfinite induction as follows.

$X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$; $X^{(\gamma)} = \bigcap_{\alpha<\gamma} X^{(\alpha)}$ for limit ordinals γ.

For a scattered space X, the smallest ordinal α such that $X^{(\alpha)} = \emptyset$ is called the *scattered height* of X and is denoted by $ht(X)$. For instance, X is discrete if and only if $ht(X) = 1$.

The following classical theorem is due to A. Pełczyński and Z. Semadeni.

Theorem 3.1. ([28, Theorem 8.5.4]) *A compact space X is scattered if and only if there is no continuous mapping of X onto the segment* $[0, 1]$.

A continuous surjection $\pi : X \to Y$ is called *irreducible* (see [28, Definition 7.1.11]) if for every closed subset F of X the condition $\pi(F) = Y$ implies $F = X$.

Proposition 3.2. ([28, Proposition 7.1.13]) *Let X be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then there exists a closed subset F of X such that $\pi(F) = Y$ and the restriction $\pi\restriction_F : F \to Y$ is irreducible.*

Proposition 3.3. ([28, Proposition 25.2.1]) *Let X be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then π is irreducible if and only if whenever $E \subset X$ and $\pi(E)$ is dense in Y, then E is dense in X.*

Recall that a Tychonoff space X is *Čech-complete* if X is a G_δ-set in some (equivalently, any) compactification of X, (see [9, 3.9.1]). It is well known that every locally compact space and every completely metrizable space is Čech-complete. Next statement resolves an open question posed in [13].

Theorem 3.4. *Every Čech-complete (in particular, compact) Δ-space is scattered.*

Proof. Step 1: X is compact. On the contrary, assume that X is not scattered. First, by Theorem 3.1, there is a continuous mapping π from X onto the segment $[0, 1]$. Second, by Proposition 3.2, there exists a closed subset F of X such that $\pi(F) = [0, 1]$ and the restriction $\pi\restriction_F : F \to [0, 1]$ is irreducible. Since $X \in \Delta$ the compact space F also belongs to Δ, by Corollary 2.2. For simplicity, without loss of generality we may assume that F is X itself and $\pi : X \to [0, 1]$ is irreducible.

Let $\{X_n : n \in \omega\}$ be a partition of $[0, 1]$ into dense sets. Put $Y_n = \bigcup_{k\geq n} X_k$, and $Z_n = \pi^{-1}(Y_n)$ for all $n \in \omega$. Then all sets Z_n are dense in X by Proposition 3.3 and the intersection $\bigcap_{n\in\omega} Z_n$ is empty. Every compact space X is a Baire space, i.e. the Baire category theorem holds in X, hence if $\{U_n : n \in \omega\}$ is any open expansion of $\{Z_n : n \in \omega\}$, then the intersection $\bigcap_{n\in\omega} U_n$ is dense in X. In view of our Theorem 2.1 this conclusion contradicts the assumption $X \in \Delta$, and the proof follows.

Step 2: X is any Čech-complete space. By the first step we deduce that every compact subset of X is scattered. But any Čech-complete space X is scattered if and only if every compact subset of X is scattered. A detailed proof of this probably folklore statement can be found in [29]. □

Proposition 3.5. *If X is a first-countable compact space, then $X \in \Delta$ if and only if X is countable.*

Proof. If $X \in \Delta$, then X is scattered, by Theorem 3.4. By the classical theorem of S. Mazurkiewicz and W. Sierpiński [28, Theorem 8.6.10], a first-countable compact space is scattered if and only if it is countable. This proves (i) ⇒ (ii). The converse is known [13] and follows from the fact that any countable space $X = \{x_n : n \in \omega\}$ admits a scant cover. Indeed, define $X_n = \{x_i : i \geq n\}$. Then the family $\{X_n : n \in \omega\}$ is a scant cover of X. Now it suffices to mention Remark 2.4. □

Remark 3.6. Theorem 3.4 extends also a well-known result of B. Knaster and K. Urbanik stating that every countable Čech-complete space is scattered [20]. It is easy to see that a countable Baire space contains a dense subset of isolated points, but in general does not have to be scattered. We don't know whether every Baire Δ-space must have isolated points.

Recall that an Eberlein compact is a compact space homeomorphic to a subset of a Banach space with the weak topology. A compact space is said to be a Corson compact space if it can be embedded in a Σ-product of the real lines. Every Eberlein compact is Corson, but not vice versa. However, every scattered Corson compact space is a scattered Eberlein compact space [1].

Theorem 3.7. ([13]) *A Corson compact space X belongs to the class Δ if and only if X is a scattered Eberlein compact space.*

Bearing in mind Theorem 3.4, to show Theorem 3.7 it suffices to use the fact that every scattered Eberlein compact space admits a scant cover (the latter follows from the proof of [4, Lemma 1.1]) and then apply Remark 2.4.

Being motivated by the previous results one can ask if there exist scattered compact spaces $X \in \Delta$ which are not Eberlein compact. The next question is also crucial: Does there exist a compact scattered space $X \notin \Delta$? Below we answer both questions positively.

We need the following somewhat technical

Theorem 3.8. *Let $Z = C_0 \cup C_1$ be a Tychonoff space such that*

(1) $C_0 \cap C_1 = \emptyset$.
(2) *C_0 is an open F_σ subset of Z.*
(3) *both C_0 and C_1 belong to the class Δ.*

Then Z also belongs to the class Δ.

Proof. By assumption, $C_0 = \bigcup\{F_n : n \in \omega\}$, where each F_n is closed in Z. Let $\{X_n : n \in \omega\}$ be any countable collection of pairwise disjoint subsets of Z. Our target is to define open sets $U_n \supseteq X_n$, $n \in \omega$ in such a way that the collection $\{U_n : n \in \omega\}$ is point-finite. We decompose the sets $X_n = X_n^0 \cup X_n^1$, where $X_n^0 = X_n \cap C_0$ and $X_n^1 = X_n \cap C_1$. By Theorem 2.1, the collection $\{X_n^0 : n \in \omega\}$ expands to a point-finite open collection $\{U_n^0 : n \in \omega\}$ in C_0. The set C_0 is open in Z, therefore U_n^0 are open in Z as well.

Now we consider the disjoint collection $\{X_n^1 : n \in \omega\}$ in C_1. By assumption, $C_1 \in \Delta$, therefore applying Theorem 2.1 once more, we find a point-finite expansion $\{V_n^1 : n \in \omega\}$ in C_1 consisting of sets which are open in C_1. Every set V_n^1 is a trace of some set W_n^1, which is open in Z, i.e. $V_n^1 = W_n^1 \cap C_1$, and every W_n^1 is open in Z. We refine the sets W_n^1 by the formula $U_n^1 = W_n^1 \setminus \bigcup\{F_i : i \leq n\}$. Since all sets F_i are closed in Z, the sets U_n^1 remain open in Z. Since all sets F_i are disjoint with C_1, the collection $\{U_n^1 : n \in \omega\}$ remains to be an expansion of $\{X_n^1 : n \in \omega\}$. Furthermore, the collection $\{U_n^1 : n \in \omega\}$ is point-finite, because $\{V_n^1 : n \in \omega\}$ is point-finite, and every point $z \in C_0$ belongs to some F_n, hence $z \notin U_m^1$ for every $m \geq n$. Finally, we define $U_n = U_n^0 \cup U_n^1$. The collection $\{U_n : n \in \omega\}$ is a point-finite open expansion of $\{X_n : n \in \omega\}$, and the proof is complete. □

This yields the following

Corollary 3.9. *Let Z be any separable scattered Tychonoff space such that its scattered height $ht(Z)$ is equal to 2. Then $Z \in \Delta$.*

Proof. The structure of Z is the following. $Z = C_0 \cup C_1$, where C_0 is a countable dense in Z set consisting of isolated in Z points and C_1 consists of all accumulation points. Moreover, the space C_1 with the topology induced from Z is discrete. All conditions of Theorem 3.8 are satisfied, and the result follows. □

Our first example will be the one-point compactification of an Isbell–Mrówka space $\Psi(\mathcal{A})$. We recall the construction and basic properties of $\Psi(\mathcal{A})$. Let $\mathcal{A}$ be an almost disjoint family of subsets of the set of natural numbers $\mathbb{N}$ and let $\Psi(\mathcal{A})$ be the set $\mathbb{N} \cup \mathcal{A}$ equipped with the topology defined as follows. For each $n \in \mathbb{N}$, the singleton $\{n\}$ is open, and for each $A \in \mathcal{A}$, a base of neighbourhoods of A is the collection of all sets of the form $\{A\} \cup B$, where $B \subset A$ and $|A \setminus B| < \omega$. The space $\Psi(\mathcal{A})$ is then a first-countable separable locally compact Tychonoff space. If $\mathcal{A}$ is a maximal almost disjoint (MAD) family, then the corresponding Isbell–Mrówka space $\Psi(\mathcal{A})$ would be in addition pseudocompact. (Readers are advised to consult [18, Chapter 8] which surveys various topological properties of these spaces).

Theorem 3.10. *There exists a separable scattered compact space X with the following properties:*

(a) *The scattered height of X is equal to 3.*
(b) *$X \in \Delta$.*
(c) *X is not an Eberlein compact space.*

Proof. Let $\mathcal{A}$ be any uncountable almost disjoint (in particular, MAD) family of subsets of $\mathbb{N}$ and let Z be the corresponding first-countable separable locally compact Isbell–Mrówka space $\Psi(\mathcal{A})$. It is easy to see that $Z = \Psi(\mathcal{A})$ satisfies the assumptions of Corollary 3.9. Hence, $Z \in \Delta$. Now, denote by X the one-point compactification of the separable locally compact space Z. Then the scattered height of X is equal to 3. Note that $X \in \Delta$ by Proposition 2.3. Moreover, X is not an Eberlein compact space, since every separable Eberlein compact space is metrizable, while $\Psi(\mathcal{A})$ is metrizable if and only if $\mathcal{A}$ is countable. □

Now we show that there exist scattered compact spaces which are not in the class Δ. We will use the classical Pressing Down Lemma. Let $[0, \omega_1)$ be the set of all countable ordinals equipped with the order topology. For simplicity, we identify $[0, \omega_1)$ with ω_1. A subset S of ω_1 is called a *stationary* subset if S has nonempty intersection with every closed and unbounded set in ω_1. A mapping $\varphi : S \to \omega_1$ is called *regressive* if $\varphi(\alpha) < \alpha$ for each $\alpha \in S$. The proof of the following fundamental statement can be found for instance in [22].

Theorem 3.11. ***Pressing Down Lemma.*** *Let $\varphi : S \to \omega_1$ be a regressive mapping, where S is a stationary subset of ω_1. Then for some $\gamma < \omega_1$, $\varphi^{-1}\{\gamma\}$ is a stationary subset of ω_1.*

It is known that there are plenty of stationary subsets of ω_1. In particular, every stationary set can be partitioned into countably many pairwise disjoint stationary sets [22]. Note that ω_1 is a scattered locally compact and first-countable space. Next statement resolves an open question posed in [13].

Theorem 3.12. *The compact scattered space* $[0, \omega_1]$ *is not in the class* Δ.

Proof. It suffices to show that ω_1 does not belong to the class Δ. Assume, on the contrary, that $\omega_1 \in \Delta$. Denote by L the set of all countable limit ordinals. Evidently, L is a closed unbounded set in ω_1. Take any representation of L as the union of countably many pairwise disjoint stationary sets $\{S_n : n \in \omega\}$. By Theorem 2.1, there exists a point-finite open expansion $\{U_n : n \in \omega\}$ in ω_1.

Fr every $\alpha \in U_n$ there is an ordinal $\beta(\alpha) < \alpha$ such that $[\beta(\alpha), \alpha] \subset U_n$. In fact, for every $n \in \omega$ we can define a regressive mapping $\varphi_n : S_n \to \omega_1$ by the formula: $\varphi_n(\alpha) = \beta(\alpha)$. Since S_n is a stationary set for every n, we can apply to φ_n the Pressing Down Lemma. Hence, for each n there are a countable ordinal γ_n and an uncountable subset $T_n \subset S_n$ with the following property: $[\gamma_n, \alpha] \subset U_n$ for every $\alpha \in T_n$. Denote $\gamma = \sup\{\gamma_n : n \in \omega\} \in \omega_1$. Because all T_n are unbounded, for all natural n we have an ordinal $\alpha_n \in T_n$ such that $\gamma < \alpha_n$ and $[\gamma_n, \alpha_n] \subset U_n$. This implies that $\gamma \in U_n$ for every $n \in \omega$. However, a collection $\{U_n : n \in \omega\}$ is point-finite. The obtained contradiction finishes the proof. □

The function space $C_k(X)$ is called *Asplund* if every separable vector subspace of $C_k(X)$ isomorphic to a Banach space, has the separable dual.

Proposition 3.13. *If a Tychonoff space* X *belongs to the class* Δ*, then the space* $C_k(X)$ *is Asplund. The converse conclusion fails in general.*

Proof. Let $\mathcal{K}(X)$ be the family of all compact subset of X. By the assumption and Corollary 2.2, each $K \in \mathcal{K}(X)$ belongs to the class Δ. Clearly, $C_k(X)$ is isomorphic to a (closed) subspace of the product $\Pi = \prod_{K \in \mathcal{K}(X)} C_k(K)$ of Banach spaces $C_k(K)$. Assume that E is a separable vector subspace of $C_k(X)$ isomorphic to a Banach space. Observe that E is isomorphic to a subspace of the finite product $\prod_{j \in F} C_k(K_j)$ for $K_j \in \mathcal{K}(X)$ and $j \in F$. Indeed, let B be the unit (bounded) ball of the normed space E. Then there exists a finite set F such that $\bigcap_{j \in F} \pi_j^{-1}(U_j) \cap \Pi \subset B$, where U_j are balls in spaces $C_k(K_j)$, $j \in F$, and π_j are natural projections from E onto $C_k(K_j)$. Let π_F be the (continuous) projection from Π onto $\prod_{j \in F} C_k(K_j)$. Then $\pi_F \restriction_E$ is an injective continuous and open map from E onto $(\pi_F \restriction_E)(E) \subset \prod_{j \in F} C_k(K_j)$. The injectivity of $\pi_F \restriction_E$ follows from the fact that B is a bounded neighbourhood of zero in E. It is easy to see that the image $(\pi_F \restriction_E)(B)$ is an open neighbourhood of zero in $\prod_{j \in F} C_k(K_j)$. On the other hand, $\prod_{j \in F} C_k(K_j)$ is isomorphic to the space $C_k(\bigoplus_{j \in F} K_j)$ and the compact space $\bigoplus_{j \in F} K_j$ is scattered. By the classical [10, Theorem 12.29] E must have the separable dual E^*. Hence, $C_k(X)$ is Asplund. The converse fails, as Theorem 3.12 shows for $X = [0, \omega_1]$. □

Since every infinite compact scattered space X contains a nontrivial converging sequence, for such X the Banach space $C(X)$ is not a Grothendieck space, (see [7]).

Corollary 3.14. *If* X *is an infinite compact and* $X \in \Delta$*, then the Banach space* $C(X)$ *is not a Grothendieck space. The converse fails, as* $X = [0, \omega_1]$ *applies.*

For non-scattered spaces X Theorem 3.4 implies immediately the following

Corollary 3.15. *If* X *is a non-scattered space, the Stone-Čech compactification* βX *is not in the class* Δ.

Proposition 3.16. *Let* $X = \beta Z \setminus Z$*, where* Z *is any infinite discrete space. Then* X *is not in the class* Δ.

Proof. $\beta Z \setminus Z$ does not have isolated points for any infinite discrete space Z. □

It is known that $X = [0, \omega_1]$ is the Stone-Čech compactification of $[0, \omega_1)$. We showed that $X \notin \Delta$. Also, $\beta Z \notin \Delta$ for any infinite discrete space Z. Every scattered Eberlein compact space belongs to the class Δ by Theorem 3.7, however no Eberlein compact X can be the Stone-Čech compactification βZ for any proper subset Z of X by the Preiss–Simon theorem (see [2, Theorem IV.5.8]). All these facts provides a motivation for the following result.

Example 3.17. There exists an Isbell–Mrówka space Z which is *almost compact* in the sense that the one-point compactification of Z coincides with βZ (see [18, Theorem 8.6.1]). Define $X = \beta Z$. Then $X \in \Delta$, by Theorem 3.10.

4. Metrizable spaces $X \in \Delta$

In this section we try to describe constructively the structure of nontrivial metrizable spaces $X \in \Delta$. Note first that every scattered metrizable X is in the class Δ since every such space X homeomorphically embeds into a scattered Eberlein compact [3], and then Theorem 3.7 and Corollary 2.2 apply. We extend this result as follows.

A topological space X is said to be *σ-scattered* if X can be represented as a countable union of scattered subspaces and X is called *strongly σ-discrete* if it is a union of countably many of its closed discrete subspaces. Strongly σ-discreteness of X implies that X is σ-scattered, for any topological space. For metrizable X, by the classical result of A. H. Stone [30], these two properties are equivalent.

Proposition 4.1. *Any σ-scattered metrizable space belongs to the class Δ.*

Proof. In view of aforementioned equivalence, every subset of X is F_σ. If every subset of X is F_σ, then $X \in \Delta$. This fact apparently is well-known (see also a comment after Claim 4.2). For the sake of completeness we include a direct argument. We show that X satisfies the condition (2) of Theorem 2.1. Let $\{X_n : n \in \omega\}$ be any countable disjoint partition of X. Denote $X_n = \bigcup\{F_{n,m} : m \in \omega\}$, where each $F_{n,m}$ is closed in X. Define open sets U_n as follows: $U_0 = X$ and $U_n = X \setminus \bigcup\{F_{k,m} : k < n, m < n\}$ for $n \geq 1$. Then $\{U_n : n \in \omega\}$ is a point-finite open expansion of $\{X_n : n \in \omega\}$ in X. □

A metrizable space A is called an *absolutely analytic* if A is homeomorphic to a Souslin subspace of a complete metric space X (of an arbitrary weight), i.e. A is expressible as $A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} A_{\sigma|n}$, where each $A_{\sigma|n}$ is a closed subset of X. It is known that every absolutely analytic metrizable space X (in particular, every Borel subspace of a complete metric space) either contains a homeomorphic copy of the Cantor set or it is strongly σ-discrete. Therefore, for absolutely analytic metrizable space X the converse is true: $X \in \Delta$ implies that X is strongly σ-discrete [13].

However, the last structural result can not be proved in general for all (separable) metrizable spaces without extra set-theoretic assumptions. Let us recall several definitions of special subsets of the real line $\mathbb{R}$ (see [23], [27]).

(a) A Q-set X is a subset of $\mathbb{R}$ such that each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.
(b) A λ-set X is a subset of $\mathbb{R}$ such that each countable $A \subset X$ is G_δ in X.

(c) A Δ-set X is a subset of $\mathbb{R}$ such that for every decreasing sequence $\{D_n : n \in \omega\}$ subsets of X with empty intersection there is a decreasing expansion $\{V_n : n \in \omega\}$ consisting of open subsets of X with empty intersection.

Claim 4.2. *The existence of an uncountable separable metrizable Δ-space is equivalent to the existence of an uncountable Δ-set.*

Proof. Note that every separable metrizable space homeomorphically embeds into a Polish space $\mathbb{R}^\omega$ and the latter space is a one-to-one continuous image of the set of irrationals $\mathbb{P}$. Therefore, if M is an uncountable separable metrizable space, then there exist an uncountable set $X \subset \mathbb{R}$ and a one-to-one continuous mapping from X onto M. It is easy to see that X is a Δ-set provided M is a Δ-space. □

Note that in the original definition of a Δ-set, G. M. Reed used G_δ-sets instead of open sets and E. van Douwen observed that these two versions are equivalent [27]. From the original definition it is obvious that each Q-set must be a Δ-set. The fact that every Δ-set is a λ-set is known as well. K. Kuratowski showed that in ZFC there exist uncountable λ-sets. The existence of an uncountable Q-set is one of the fundamental set-theoretical problems considered by many authors. F. Hausdorff showed that the cardinality of an uncountable Q-set X has to be strictly smaller than the continuum $\mathfrak{c} = 2^{\aleph_0}$, so in models of ZFC plus the Continuum Hypothesis (CH) there are no uncountable Q-sets. Let us outline several known most relevant facts.

(1) Martin's Axiom plus the negation of the Continuum Hypothesis (MA $+\neg$CH) implies that every subset $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a Q-set (see [15]).

(2) It is consistent that there is a Q-set X such that its square X^2 is not a Q-set [14].

(3) The existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set, i.e. a Q-set all finite powers of which are Q-sets [25].

(4) No Δ-set X can have cardinality $\mathfrak{c}$ [26]. Hence, under MA, every subset of $\mathbb{R}$ that is a Δ-set is also a Q-set. Recently we proved the following claim: If X has a countable network and $|X| = \mathfrak{c}$, then $C_p(X)$ is not distinguished [13]. In view of our Theorem 2.1 this fact means that no Δ-space X with a countable network can have cardinality $\mathfrak{c}$. [2]

(5) It is consistent that there exists a Δ-set X that is not a Q-set [21]. Of course, there are plenty of nonmetrizable Δ-spaces with non-G_δ subsets, in ZFC.

(6) An uncountable Δ-set exists if and only if there exists a separable countably paracompact nonnormal Moore space (see [16] and [26]).

Summarizing, the following conclusion is an immediate consequence of our Theorem 2.1 and the known facts about Δ-sets listed above.

Corollary 4.3.

(1) *The existence of an uncountable separable metrizable space such that $C_p(X)$ is distinguished, is independent of ZFC.*

(2) *There exists an uncountable separable metrizable space X such that $C_p(X)$ is distinguished, if and only if there exists a separable countably paracompact nonnormal Moore space.*

[2] The referee kindly informed the authors that the last result can be derived easily from the actual argument of [26].

5. Basic operations in Δ and open problems

In this section we consider the question whether the class Δ is invariant under the following basic topological operations: subspaces, continuous images, quotient continuous images, finite/countable unions, finite products.

1. Subspaces. Trivial because of Corollary 2.2.

2. (Quotient) continuous images. Evidently, every topological space is a continuous image of a discrete one. The following assertion is a consequence of a known fact about MAD families (see [18, Chapter 8]).

Proposition 5.1. *There exists a first-countable separable pseudocompact locally compact Isbell–Mrówka space $\Psi(\mathcal{A})$ which admits a continuous surjection onto the segment* $[0,1]$.

Thus, the class Δ is not invariant under continuous images even for separable locally compact spaces. However, one can show that every uncountable quotient continuous image of any Isbell–Mrówka space $\Psi(\mathcal{A})$ satisfies the conditions of Corollary 3.9, therefore it is a Δ-space. Note also that a class of scattered Eberlein compact spaces preserves continuous images. We were unable to resolve the following major open problem.

Problem 5.2. *Let X be any compact Δ-space and Y be a continuous image of X. Is Y a Δ-space?*

Even a more general question is open.

Problem 5.3. *Let X be any Δ-space and Y be a quotient continuous image of X. Is Y a Δ-space?*

Towards a solution of these problems we obtained several partial positive results.

Proposition 5.4. *Let X be any Δ-space and $\varphi : X \to Y$ be a quotient continuous surjection with only finitely many nontrivial fibers. Then Y is also a Δ-space.*

Proof. By assumption, there exists a closed subset $K \subset X$ such that $\varphi(K)$ is finite and $\varphi \restriction_{X\setminus K}: X \setminus K \to Y \setminus \varphi(K)$ is a one-to-one mapping. Both sets $X \setminus K$ and $Y \setminus \varphi(K)$ are open in X and Y, respectively. Since φ is a quotient continuous mapping, it is easy to see that $\varphi \restriction_{X\setminus K}$ is a homeomorphism. $X \setminus K$ is a Δ-space, hence $Y \setminus \varphi(K)$ is also a Δ-space. Finally, Y is a Δ-space, by Proposition 2.3. $\square$

Proposition 5.5. *Let X be any Δ-space and $\varphi : X \to Y$ be a closed continuous surjection with finite fibers. Then Y is also a Δ-space.*

Proof. Let $\{Y_n : n \in \omega\}$ be a partition of Y. By assumption, the partition $\{\varphi^{-1}(Y_n) : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in X. Clearly, $\varphi(X \setminus U_n)$ are closed sets in Y. Define $V_n = Y \setminus \varphi(X \setminus U_n)$ for each $n \in \omega$. We have that $\{V_n : n \in \omega\}$ is an open expansion of $\{Y_n : n \in \omega\}$ in Y. It remains to verify that the family $\{V_n : n \in \omega\}$ is point-finite. Indeed, let $y \in Y$ be any point. Each point in the fiber $\varphi^{-1}(y)$ belongs to a finite number of sets U_n. Since the fiber $\varphi^{-1}(y)$ is finite, y is contained only in a finite number of sets V_n which finishes the proof. $\square$

3. Finite/countable unions.

Proposition 5.6. *Assume that X is a finite union of closed subsets X_i, where each X_i belongs to the class Δ. Then X also belongs to Δ. In particular, a finite union of compact Δ-spaces is also a Δ-space.*

Proof. Denote by Z the discrete finite union of Δ-spaces X_i. Obviously, Z is a Δ-space which admits a natural closed continuous mapping onto X. Since all fibers of this mapping are finite, the result follows from Proposition 5.5. □

We recall a definition of the Michael line. The Michael line X is the refinement of the real line $\mathbb{R}$ obtained by isolating all irrational points. So, X can be represented as a countable disjoint union of singletons (rationals) and an open discrete set. Nevertheless, the Michael line X is not in Δ [13]. This example and Proposition 5.6 justify the following

Problem 5.7. *Let X be a countable union of compact subspaces X_i such that each X_i belongs to the class Δ. Does X belong to the class Δ?*

4. Finite products. We already mentioned earlier that the existence of a Q-set $X \subset \mathbb{R}$ such that its square X^2 is not a Q-set, is consistent with ZFC.

Problem 5.8. *Is the existence of a Δ-set $X \subset \mathbb{R}$ such that its square X^2 is not a Δ-set, consistent with ZFC?*

It is known that the finite product of scattered Eberlein compact spaces is a scattered Eberlein compact.

Problem 5.9. *Let X be the product of two compact spaces X_1 and X_2 such that each X_i belongs to the class Δ. Does X belong to the class Δ?*

Our last problem is inspired by Theorem 3.10.

Problem 5.10. *Let X be any scattered compact space with a finite scattered height. Does X belong to the class Δ?*

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References

[1] K. Alster, *Some remarks on Eberlein compacts*, Fund. Math. **104** (1979), 43–46.

[2] A. V. Arkhangel'skiĭ, *Topological Function Spaces*, Kluwer, Dordrecht, 1992.

[3] T. Banakh and A. Leiderman, *Uniform Eberlein compactifications of metrizable spaces*, Topology and Appl. **159** (2012), 1691–1694.

[4] M. Bell and W. Marciszewski, *On scattered Eberlein compact spaces*, Israel J. Math. **158** (2007), 217–224.

[5] D. Bierstedt and J. Bonet, *Density conditions in Fréchet and (DF)-spaces*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 59–75.

[6] D. Bierstedt and J. Bonet, *Some aspects of the modern theory of Fréchet spaces*, RACSAM **97** (2003), 159–188.

[7] H. D. Dales, F. H. Dashiell Jr., A.T. -M. Lau, D. Strauss, *Banach Spaces of Continuous Functions as Dual Spaces*, CMS Books in Mathematics, Springer, 2016.

[8] J. Dieudonné, L. Schwartz, *La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$*, Ann. Inst. Fourier (Grenoble) **1** (1949), 61–101.

[9] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[10] M. Fabian, P. Habala, P. Hajek, V. Montesinos Santalucia, J. Pelant, V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books in Mathematics, Springer, 2001.

[11] J. C. Ferrando, J. Kąkol, *Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces*, J. Convex Anal. **26** (2019), 1337–1346.

[12] J. C. Ferrando, J. Kąkol, S. A. Saxon, *Examples of nondistinguished function dpaces $C_p(X)$*, J. Convex. Anal. **26** (2019), 1347–1348.
[13] J. C. Ferrando, J. Kąkol, A. Leiderman, S. A. Saxon, *Distinguished $C_p(X)$ spaces*, RACSAM 115:27 (2021).
[14] W. G. Fleissner, *Squares of Q-sets*, Fund. Math. **118** (1983), 223–231.
[15] W. G. Fleissner and A. W. Miller, *On Q-sets*, Proc. Amer. Math. Soc. **78** (1980), 280–284.
[16] W. G. Fleissner, G. M. Reed and M. Wage, *Normality versus countable paracompactness in perfect spaces*, Bull. Amer. Math. Soc. **82** (1976), 635–639.
[17] A. Grothendieck, *Sur les espaces (F) et (DF)*, Summa Brasil. Math. **3** (1954), 57–123.
[18] M. Hrušák, Á. Tamariz-Mascarúa, M. Tkachenko (Editors), *Pseudocompact Topological Spaces, A Survey of Classic and New Results with Open Problems*, Springer, 2018.
[19] G. Köthe, *Topological Vector Spaces, I*, Springer-Verlag, Berlin, 1983.
[20] B. Knaster, K. Urbanik, *Sur les espaces complets séparables de dimension* 0, Fund. Math. **40** (1953), 194–202.
[21] R. W. Knight, Δ-*Sets*, Trans. Amer. Math. Soc. **339** (1993), 45–60.
[22] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North-Holland, New York, 1980.
[23] A. W. Miller, *Special subsets of the real line*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 201–233.
[24] P. J. Nyikos, *A history of the normal Moore space problem*, in: Handbook of the History of General Topology, Volume 3, Kluwer, Dordrecht, 2001, 1179–1212.
[25] T. C. Przymusiński, *The existence of Q-sets is equivalent to the existence of strong Q-sets*, Proc. Amer. Math. Soc. **79** (1980), 626–628.
[26] T. C. Przymusiński, *Normality and separability of Moore spaces*, in: Set-Theoretic Topology, Academic Press, New York, 1977, 325–337.
[27] G. M. Reed, *On normality and countable paracompactness*, Fund. Math. **110** (1980), 145–152.
[28] Z. Semadeni, *Banach Spaces of Continuous Functions, Volume I*, PWN - Polish Scientific Publishers, Warszawa, 1971.
[29] D. B. Shakhmatov, M. G. Tkachenko, V. V. Tkachuk, S. Watson, and R. G. Wilson, *Neither first countable nor Čech-complete spaces are maximal Tychonoff connected*, Proc. Amer. Math. Soc. **126** (1998), 279–287.
[30] A. H. Stone, *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. **107** (1963), 58–70.

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