

Polynomial time algorithms for inclusion and equivalence of deterministic omega acceptors^{*}

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Abstract. The class of omega languages recognized by deterministic parity acceptors (DPAs) or deterministic Muller acceptors (DMAs) is exactly the regular omega languages. The inclusion problem is the following: given two acceptors \mathcal{A}_1 and \mathcal{A}_2 , determine whether the language recognized by \mathcal{A}_1 is a subset of the language recognized by \mathcal{A}_2 , and if not, return an ultimately periodic omega word accepted by \mathcal{A}_1 but not \mathcal{A}_2 . We describe polynomial time algorithms to solve this problem for two DPAs and for two DMAs. Corollaries include polynomial time algorithms to solve the equivalence problem for DPAs and DMAs, and also the inclusion and equivalence problems for deterministic Büchi and coBüchi acceptors.

1 Preliminaries

We denote the set of nonnegative integers by \mathbb{N} , and for any $m, n \in \mathbb{N}$, we let $[m..n] = \{j \in \mathbb{N} \mid m \leq j \wedge j \leq n\}$.

1.1 Words and Automata

If Σ is a finite alphabet of symbols, then Σ^* denotes the set of all finite words over Σ , ε denotes the empty word, $|u|$ denotes the length of the finite word u , and Σ^+ denotes the set of all nonempty finite words over Σ . Also, Σ^ω denotes the set of all infinite words over Σ , termed ω -words. For a finite or infinite word w , the successive symbols are indexed by positive integers, and $w[i]$ denotes the symbol with index i . For words $u \in \Sigma^*$ and $v \in \Sigma^+$, $u(v)^\omega$ denotes the ultimately periodic ω -word consisting of u followed by infinitely many copies of v .

A *complete deterministic automaton* is a tuple $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$ consisting of a finite alphabet Σ of symbols, a finite set Q of states, an initial state $q_i \in Q$, and a transition function $\delta : Q \times \Sigma \rightarrow Q$. In this paper, *automaton* will mean a complete deterministic automaton. We extend δ to the domain $Q \times \Sigma^*$ inductively in the usual way, and for $u \in \Sigma^*$, define $\mathcal{M}(u) = \delta(q_i, u)$, the state of \mathcal{M} reached on input u . Also, for a state $q \in Q$, \mathcal{M}^q denotes the automaton $\langle \Sigma, Q, q, \delta \rangle$, in which the initial state has been changed to q .

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The *run* of an automaton \mathcal{M} on an input $u \in \Sigma^*$ is the sequence of states q_0, q_1, \dots, q_k , where $k = |u|$, $q_0 = q_\iota$, and for each $i \in [1..k]$, $q_i = \delta(q_{i-1}, u[i])$. The *run* of \mathcal{M} on an input $w \in \Sigma^\omega$ is the infinite sequence of states q_0, q_1, q_2, \dots , where $q_0 = q_\iota$, and for each positive integer i , $q_i = \delta(q_{i-1}, w[i])$. For an infinite word $w \in \Sigma^\omega$, let $\text{Inf}_{\mathcal{M}}(w)$ denote the set of states of \mathcal{M} that appear infinitely often in the run of \mathcal{M} on input w .

A state q of an automaton \mathcal{M} is *reachable* if and only if there exists a finite word $u \in \Sigma^*$ such that $\mathcal{M}(u) = q$. We may restrict \mathcal{M} to contain only its reachable states without affecting its finite or infinite runs.

For any automaton $\mathcal{M} = \langle \Sigma, Q, q_\iota, \delta \rangle$ we may construct a related directed graph $G(\mathcal{M}) = (V, E)$ as follows. The set V of vertices is just the set of states Q , and there is a directed edge $(q_1, q_2) \in E$ if and only if for some symbol $\sigma \in \Sigma$ we have $\delta(q_1, \sigma) = q_2$. By processing every pair $(q, \sigma) \in Q \times \Sigma$, the set of edges $(q_1, q_2) \in E$ may be constructed in time $O(|\Sigma| \cdot |Q|)$ using a hash table representation of E .

There are some differences in the terminology related to strong connectivity between graph theory and omega automata, which we resolve as follows. In graph theory, a *path* of length k from u to v in a directed graph (V, E) is a finite sequence of vertices v_0, v_1, \dots, v_k such that $u = v_0$, $v = v_k$ and for each i with $i \in [1..k]$, $(v_{i-1}, v_i) \in E$. Thus, for every vertex v , there is a path of length 0 from v to v . A set of vertices S is *strongly connected* if and only if for all $u, v \in S$, there is a path of some nonnegative length from u to v and all the vertices in the path are elements of S . Thus, for every vertex v , the singleton set $\{v\}$ is a strongly connected set of vertices. A *strongly connected component* of a directed graph is a maximal strongly connected set of vertices. There is a linear time algorithm to find the set of strong components of a directed graph [6].

In the theory of omega automata, a *strongly connected component* (SCC) of A is a nonempty set $C \subseteq Q$ of states such that for any $q_1, q_2 \in C$, there exists a nonempty word u such that $\delta(q_1, u) = q_2$, and for every prefix u' of u , $\delta(q_1, u') \in C$. Note that a SCC of A need not be maximal, and that a single state q of A is not a SCC of A unless for some symbol $\sigma \in \Sigma$ we have $\delta(q, \sigma) = q$.

In this paper, we use the terminology *SCC* and *maximal SCC* to refer to the definitions from the theory of omega automata, and the terminology *graph theoretic strongly connected components* to refer to the definitions from graph theory. Additionally, we use the term *trivial strong component* to refer to a graph theoretic strongly connected component that is a singleton vertex $\{v\}$ such that there is no edge (v, v) . Then if \mathcal{M} is an automaton, the maximal SCCs of \mathcal{M} are the graph theoretic strongly connected components of $G(\mathcal{M})$ with the exception of the trivial strong components. The following is a direct consequence of the definitions.

Claim 1. *For any automaton \mathcal{M} and any $w \in \Sigma^\omega$, $\text{Inf}_{\mathcal{M}}(w)$ is a SCC of \mathcal{M} .*

The Product of Two Automata. Suppose \mathcal{M}_1 and \mathcal{M}_2 are automata with the same alphabet Σ , where for $i = 1, 2$, $\mathcal{M}_i = \langle \Sigma, Q_i, (q_\iota)_i, \delta_i \rangle$. Their product automaton, denoted $\mathcal{M}_1 \times \mathcal{M}_2$, is the deterministic automaton $\mathcal{M} = \langle \Sigma, Q, q_\iota, \delta \rangle$

such that $Q = Q_1 \times Q_2$, the set of ordered pairs of states of \mathcal{M}_1 and \mathcal{M}_2 , $q_i = ((q_i)_1, (q_i)_2)$, the pair of initial states of the two automata, and for all $(q_1, q_2) \in Q$ and $\sigma \in \Sigma$, $\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))$. For $i = 1, 2$, let π_i be projection onto the i th coordinate, so that for a subset S of Q , $\pi_1(S) = \{q_1 \mid \exists q_2 (q_1, q_2) \in S\}$, and analogously for π_2 .

1.2 Acceptors

If $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$ is an automaton, we may augment it with an acceptance condition α to get a *complete deterministic acceptor* $\mathcal{A} = \langle \Sigma, Q, q_i, \delta, \alpha \rangle$, which is a machine that accepts some words and rejects others. In this paper, an *acceptor* will mean a complete deterministic acceptor. If $q \in Q$, we use the notation \mathcal{A}^q for the acceptor $\langle \Sigma, Q, q, \delta, \alpha \rangle$, in which the initial state has been changed to the state q . An acceptor accepts a word if the run on that word is accepting, as defined below for the types of acceptors we consider. For finite words the acceptance condition is a set $F \subseteq Q$ and the run on a word $v \in \Sigma^*$ is accepting iff it ends in an accepting state, that is, $\mathcal{M}(v) \in F$. We use DFA to denote the class of acceptors of finite words, and DFA for the languages they accept, which is the class of regular languages.

For ω -words w , there are various acceptance conditions in the literature; we consider four of them: Büchi, coBüchi, parity, and Muller, which are all based on $\text{Inf}_{\mathcal{M}}(w)$, the set of states visited infinitely often in the run of the automaton on the input w .

The Büchi and coBüchi acceptance conditions are also specified by a set $F \subseteq Q$. The run of a Büchi acceptor on an input word $w \in \Sigma^\omega$ is accepting iff it visits at least one state in F infinitely often, that is, $\text{Inf}_{\mathcal{M}}(w) \cap F \neq \emptyset$. The run of a coBüchi acceptor on an input word $w \in \Sigma^\omega$ is accepting iff it visits F only finitely many times, that is, $\text{Inf}_{\mathcal{M}}(w) \cap F = \emptyset$.

A parity acceptance condition is a map $\kappa : Q \rightarrow \mathbb{N}$ assigning to each state a natural number termed a color (or priority). We extend κ to sets of states in the natural way, that is, for $S \subseteq Q$, $\kappa(S) = \{\kappa(q) \mid q \in S\}$. For a parity acceptor \mathcal{P} and an ω -word w , we denote by $\mathcal{P}(w)$ the minimum color of all states visited infinitely often by \mathcal{P} on input w , that is,

$$\mathcal{P}(w) = \min(\kappa(\text{Inf}_{\mathcal{M}}(w))).$$

The run of a parity acceptor \mathcal{P} on an input word $w \in \Sigma^\omega$ is accepting iff the minimum color visited infinitely often is odd, that is, $\mathcal{P}(w)$ is odd.

A Muller acceptance condition is a family of final state sets $\mathcal{F} = \{F_1, \dots, F_k\}$ for some $k \in \mathbb{N}$ and $F_i \subseteq Q$ for $i \in [1..k]$. The run of a Muller acceptor is accepting iff the set of states visited infinitely often in the run is an element of \mathcal{F} , that is, $\text{Inf}_{\mathcal{M}}(w) \in \mathcal{F}$.

To measure the running times of algorithms taking acceptors as inputs, we specify size measures for these acceptors. For an automaton $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$, we specify its size as $|Q| \cdot |\Sigma|$, which is the number of entries in a table that explicitly specifies the transition function δ . Note that the size of the product

automaton $\mathcal{M}_1 \times \mathcal{M}_2$ is at most $|\Sigma| \cdot |Q_1| \cdot |Q_2|$, where for $i = 1, 2$, Q_i is the set of states of \mathcal{M}_i . For a deterministic Büchi, coBüchi, or parity acceptor, the space to specify the acceptance condition is dominated by the size of the automaton, so that is the size of the acceptor. However, for deterministic Muller acceptors, we add the quantity $|Q| \cdot |\mathcal{F}|$, which bounds the size of a table explicitly specifying the membership of each state in each final state set in \mathcal{F} .³

We use $[[\mathcal{A}]]$ to denote the set of words accepted by a given acceptor \mathcal{A} . Two acceptors \mathcal{A}_1 and \mathcal{A}_2 are *equivalent* iff $[[\mathcal{A}_1]] = [[\mathcal{A}_2]]$. We use DBA, DCA, DPA, and DMA for the classes of (deterministic) Büchi, coBüchi, parity, and Muller acceptors. We use $\mathbb{D}\text{BA}$, $\mathbb{D}\text{CA}$, $\mathbb{D}\text{PA}$, and $\mathbb{D}\text{MA}$ for the classes of languages they recognize, respectively. $\mathbb{D}\text{PA}$ and $\mathbb{D}\text{MA}$ are each the full class of regular ω -languages, while $\mathbb{D}\text{BA}$ and $\mathbb{D}\text{CA}$ are proper subclasses.

We observe the following known facts about the relationships between DBAs, DCAs and DPAs.

Claim 2. *Let $\mathcal{B} = \mathcal{C} = \langle \Sigma, Q, q_i, \delta, F \rangle$, where \mathcal{B} is a DBA and \mathcal{C} is a DCA. Then the languages recognized by \mathcal{B} and \mathcal{C} are complements of each other, that is, $[[\mathcal{B}]] = \Sigma^\omega \setminus [[\mathcal{C}]]$.*

Claim 3. *Let the DBA $\mathcal{B} = \langle \Sigma, Q, q_i, \delta, F \rangle$. Define the coloring $\kappa_{\mathcal{B}}(q) = 1$ for all $q \in F$ and $\kappa_{\mathcal{B}}(q) = 2$ for all $q \in (Q \setminus F)$. Define the DPA $\mathcal{P} = \langle \Sigma, Q, q_i, \delta, \kappa_{\mathcal{B}} \rangle$. Then \mathcal{B} and \mathcal{P} accept the same language, that is, $[[\mathcal{B}]] = [[\mathcal{P}]]$.*

Analogously, for DCAs we have the following.

Claim 4. *Let the DCA $\mathcal{B} = \langle \Sigma, Q, q_i, \delta, F \rangle$. Define the coloring $\kappa_{\mathcal{C}}(q) = 0$ for all $q \in F$ and $\kappa_{\mathcal{C}}(q) = 1$ for all $q \in (Q \setminus F)$. Define the DPA $\mathcal{P} = \langle \Sigma, Q, q_i, \delta, \kappa_{\mathcal{C}} \rangle$. Then \mathcal{C} and \mathcal{P} accept the same language, that is, $[[\mathcal{C}]] = [[\mathcal{P}]]$.*

1.3 Right congruences

An equivalence relation \sim on Σ^* is a *right congruence* if $x \sim y$ implies $xv \sim yv$ for every $x, y, v \in \Sigma^*$. The *index* of \sim , denoted $|\sim|$ is the number of equivalence classes of \sim .

Given an automaton $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$, we can associate with it a right congruence as follows: $x \sim_{\mathcal{M}} y$ iff \mathcal{M} reaches the same state when reading x or y , that is, $\mathcal{M}(x) = \mathcal{M}(y)$. If all the states of \mathcal{M} are reachable then the index of $\sim_{\mathcal{M}}$ is exactly the number of states of \mathcal{M} .

Given a language $L \subseteq \Sigma^*$, its *canonical right congruence* \sim_L is defined as follows: $x \sim_L y$ iff $\forall z \in \Sigma^*$ we have $xz \in L \iff yz \in L$. For a word $v \in \Sigma^*$, the notation $[v]$ is used for the equivalence class of \sim in which v resides.

With a right congruence \sim of finite index one can naturally associate an automaton $\mathcal{M}_{\sim} = \langle \Sigma, Q, q_i, \delta \rangle$ as follows. The set of states Q consists of the equivalence classes of \sim . The initial state q_i is the equivalence class $[\varepsilon]$. The transition function δ is defined by $\delta([u], a) = [ua]$.

³ See [2] for a discussion regarding size of Muller automata in the literature.

The Myhill-Nerode Theorem states that a language $L \subseteq \Sigma^*$ is regular iff \sim_L is of finite index. Moreover, if L is accepted by a DFA \mathcal{A} with automaton \mathcal{M} , then $\sim_{\mathcal{M}}$ refines \sim_L . Finally, the index of \sim_L gives the number of states of the minimal DFA for L .

For an ω -language $L \subseteq \Sigma^\omega$, the right congruence \sim_L is defined analogously, by quantifying over ω -words. That is, $x \sim_L y$ iff $\forall z \in \Sigma^\omega$ we have $xz \in L \iff yz \in L$.

For a regular ω -language L , the right congruence relation \sim_L is always of finite index, and we may define the *right congruence automaton* for L to be the automaton M_{\sim_L} . However, in contrast to the Myhill-Nerode Theorem, the right congruence automaton for L may not be adequate to support an acceptor for L . As an example consider the language $L = (a + b)^*(bba)^\omega$. We have that \sim_L consists of just one equivalence class, since for any $x \in \Sigma^*$ and $w \in \Sigma^\omega$ we have that $xw \in L$ iff w has $(bba)^\omega$ as a suffix. However, a DPA or DMA recognizing L needs more than a single state.

The classes \mathbb{B} , \mathbb{C} , \mathbb{P} , \mathbb{M} of omega languages are defined as those for which the right congruence automaton is adequate to support an acceptor of the corresponding type. A language L is in \mathbb{B} (resp., \mathbb{C} , \mathbb{P} , \mathbb{M}) if there exists a DBA (resp., DCA, DPA, DMA) \mathcal{A} such that $L = \llbracket \mathcal{A} \rrbracket$ and the automaton part of \mathcal{A} is isomorphic to the right congruence automaton of L . These classes are more expressive than one might conjecture; it was shown in [1] that in every class of the infinite Wagner hierarchy [8] there are languages in \mathbb{P} .

1.4 The inclusion and equivalence problems

The *inclusion problem* for two ω -acceptors is the following. Given as input two ω -acceptors \mathcal{A}_1 and \mathcal{A}_2 over the same alphabet, determine whether the language accepted by \mathcal{A}_1 is a subset of the language accepted by \mathcal{A}_2 , that is, whether $\llbracket \mathcal{A}_1 \rrbracket \subseteq \llbracket \mathcal{A}_2 \rrbracket$. If so, the answer should be “yes”; if not, the answer should be “no” and a *witness*, that is, an ultimately periodic ω -word $u(v)^\omega$ accepted by \mathcal{A}_1 but rejected by \mathcal{A}_2 .

The *equivalence problem* for two ω -acceptors is similar: the input is two ω -acceptors \mathcal{A}_1 and \mathcal{A}_2 over the same alphabet, and the problem is to determine whether they are equivalent, that is, whether $\llbracket \mathcal{A}_1 \rrbracket = \llbracket \mathcal{A}_2 \rrbracket$. If so, the answer should be “yes”; if not, the answer should be “no” and a witness, that is, an ultimately periodic ω -word $u(v)^\omega$ that is accepted by one of the two acceptors and not accepted by the other.

Clearly, if we have a procedure to solve the inclusion problem, at most two calls to it will solve the equivalence problem. Thus, we focus on the inclusion problem. By Claim 2, the inclusion and equivalence problems for DCAs are efficiently reducible to those for DBAs, and vice versa. Also, by Claims 3 and 4, the inclusion and equivalence problems for DBAs and DCAs are efficiently reducible to those for DPAs. Hence we consider the problems of inclusion for two DPAs and inclusion for two DMAs.

Note that while a polynomial algorithm for testing inclusion of DFAs can be obtained using polynomial algorithms for complementation, intersection and

emptiness (since for any two languages $L_1 \subseteq L_2$ if and only if $L_1 \cap \overline{L_2} = \emptyset$), a similar approach does not work in the case of DPAs; although complementation and emptiness for DPAs can be computed in polynomial time, intersection cannot [3, Theorem 9].

For the inclusion problem for DBAs, DCAs, and DPAs, Schewe [5] gives the following result.

Theorem 5. *The inclusion problems for DBAs, DCAs, and DPAs are in NL.*

Because NL (nondeterministic logarithmic space) is contained in polynomial time, this implies the existence of polynomial time inclusion and equivalence algorithms for DBAs, DCAs, and DPAs. Because Schewe does not give a proof or reference for Theorem 5, for completeness we include a brief proof.

Proof. For $i = 1, 2$, let $\mathcal{P}_i = \langle \Sigma, Q_i, (q_i)_i, \delta_i, \kappa_i \rangle$ be a DPA. It suffices to guess two states $q_1 \in Q_1$ and $q_2 \in Q_2$, and two words $u \in \Sigma^*$ and $v \in \Sigma^+$, and to check that for $i = 1, 2$, $\delta_i((q_i)_i, u) = q_i$ and $\delta_i(q_i, v) = q_i$, and also, that the smallest value of $\kappa_1(q)$ in the loop in \mathcal{P}_1 from q_1 to q_1 on input v is odd, while the smallest value of $\kappa_2(q)$ in the loop in \mathcal{P}_2 from q_2 to q_2 on input v is even. Logarithmic space is enough to record the two guessed states q_1 and q_2 as well as the current minimum values of κ_1 and κ_2 as the loops on v are traversed in the two automata. The words u and v need only be guessed symbol-by-symbol, using a pointer in each automaton to keep track of its current state. \square

This approach does not seem to work in the case of testing DMA inclusion, because the acceptance criterion for DMAs would require keeping track of the set of states traversed in the loop on the word v , which would in general require more than logarithmic space. In this paper, we give an explicit polynomial time algorithm for testing DPA inclusion, as well as a polynomial time algorithm for testing DMA inclusion, a novel result.

2 An inclusion algorithm for DPAs

In this section we describe an explicit polynomial time algorithm for the inclusion problem for two DPAs.

2.1 Constructing a witness

In order to return a witness, the inclusion algorithm calls a procedure **Witness** that takes as input an automaton $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$ with no unreachable states and a SCC C of \mathcal{M} and returns an ultimately periodic word $u(v)^\omega$ such that $\text{Inf}_{\mathcal{M}}(u(v)^\omega) = C$. The procedure first chooses a state $q \in C$ and uses breadth first search in $G(\mathcal{M})$ to find a shortest finite word u such that $\mathcal{M}(u) = q$. The length of u is at most $|Q| - 1$.

If $|C| = 1$, then the procedure finds a symbol $\sigma \in \Sigma$ such that $\delta(q, \sigma) = q$, and returns $u(\sigma)^\omega$. Otherwise, for every $q' \in C$ such that $q' \neq q$, it uses breadth

first search in the subgraph of $G(\mathcal{M})$ induced by the vertices in C to find shortest finite words $v_{q,q'}$ and $v_{q',q}$ such that $\delta(q, v_{q,q'}) = q'$ and $\delta(q', v_{q',q}) = q$, and no intermediate state is outside of C . The finite word v is the concatenation, over all $q' \in C$ with $q' \neq q$ of the finite words $v_{q,q'} \cdot v_{q',q}$. The length of v is at most $O(|C|^2)$.

In \mathcal{M} , the input u reaches q , and from q the word v visits no state outside of C , visits each state of C , and returns to q . Thus, $\text{Inf}_{\mathcal{M}}(u(v)^\omega) = C$, as desired. The length of $u(v)^\omega$ is bounded by $O(|Q| + |C|^2)$. We have proved the following.

Lemma 1. *The **Witness** procedure takes as input an automaton \mathcal{M} with no unreachable states and a SCC C of \mathcal{M} , and returns an ultimately periodic word $u(v)^\omega$ such that $\text{Inf}_{\mathcal{M}}(u(v)^\omega) = C$.*

*The length of $u(v)^\omega$ is bounded by $O(|Q| + |C|^2)$, where Q is the set of states of \mathcal{M} . The running time of the **Witness** procedure is bounded by this quantity plus $O(|Q| \cdot |\Sigma|)$.*

2.2 Searching for w with $\mathcal{P}_1(w) = k_1$ and $\mathcal{P}_2(w) = k_2$

We next describe a procedure **Colors** that takes as input two DPAs over the same alphabet, $\mathcal{P}_i = \langle \Sigma, Q_i, (q_i)_i, \delta_i, \kappa_i \rangle$ for $i = 1, 2$, and two nonnegative integers k_1 and k_2 , and answers the question of whether there exists an ω -word w such that for $i = 1, 2$, $\mathcal{P}_i(w) = k_i$, that is, the minimum color of the states visited infinitely often on input w in \mathcal{P}_1 is k_1 and in \mathcal{P}_2 is k_2 . If there is no such w , the return value will be “no”, but if there is such a w , the return value will be “yes” and an ultimately periodic witness $u(v)^\omega$ such that $\mathcal{P}_1(u(v)^\omega) = k_1$ and $\mathcal{P}_2(u(v)^\omega) = k_2$.

For $i = 1, 2$, let \mathcal{M}_i be the automaton of \mathcal{P}_i , that is, $\mathcal{M}_i = \langle \Sigma, Q_i, (q_i)_i, \delta_i \rangle$. The **Colors** procedure constructs the product automaton $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, eliminating unreachable states. It then constructs the related directed graph $G(\mathcal{M})$, and the subgraph G' of $G(\mathcal{M})$ obtained by removing all vertices (q_1, q_2) such that $\kappa_1(q_1) < k_1$ or $\kappa_2(q_2) < k_2$ and their incident edges.

In linear time in the size of G' , the **Colors** procedure computes the graph theoretic strongly connected components of G' and eliminates any trivial strong components. The procedure then loops through the nontrivial strong components C of G' checking whether $\min(\kappa_1(\pi_1(C))) = k_1$ and $\min(\kappa_2(\pi_2(C))) = k_2$. If so, the **Witness** procedure is called with inputs \mathcal{M} and C , and the resulting witness $u(v)^\omega$ is returned with the answer “yes”. If none of the nontrivial strong components of G' satisfies this condition, then the answer “no” is returned.

Theorem 6. *The **Colors** procedure takes as input two arbitrary DPAs \mathcal{P}_1 and \mathcal{P}_2 over the same alphabet Σ and two nonnegative integers k_1 and k_2 , and determines whether there exists an ω -word w such that for $i = 1, 2$, $\mathcal{P}_i(w) = k_i$, returning the answer “no” if not, and returning the answer “yes” and a witness word $w = u(v)^\omega$ such that $\mathcal{P}_i(u(v)^\omega) = k_i$ for $i = 1, 2$, if so.*

*The length of a witness $u(v)^\omega$ is bounded by $O((|Q_1| \cdot |Q_2|)^2)$ and the running time of the **Colors** procedure is bounded by this quantity plus $O(|\Sigma| \cdot |Q_1| \cdot |Q_2|)$, where Q_i is the set of states of \mathcal{P}_i for $i = 1, 2$.*

Proof. Assume first that the **Colors** procedure returns “yes” with a witness $u(v)^\omega$. This occurs only if the procedure finds a nontrivial graph theoretic strong component C of G' such that $\min(\kappa_1(\pi_1(C))) = k_1$ and $\min(\kappa_2(\pi_2(C))) = k_2$. For $i = 1, 2$, $\pi_i(C)$ is the set of states visited infinitely often by \mathcal{M}_i on the input $u(v)^\omega$, which has minimum color k_i . That is, for $i = 1, 2$, $\mathcal{P}_i(u(v)^\omega) = k_i$, and $u(v)^\omega$ is a correct witness for the answer “yes”.

To see that the **Colors** procedure does not incorrectly answer “no”, we argue as follows. Suppose w is an ω -word such that for $i = 1, 2$, $\mathcal{P}_i(w) = k_i$, that is, if $C_i = \text{Inf}_{\mathcal{M}_i}(w)$ then $\min(\kappa_i(C_i)) = k_i$. Clearly, no state in C_i has a color less than k_i , so if $C = \text{Inf}_{\mathcal{M}}(w)$ is the set of states visited infinitely often in \mathcal{M} on input w , all the elements of C will be in G' . Because C is a SCC of \mathcal{M} , C is a nontrivial graph theoretic strongly connected set of vertices of G' . Thus C is contained in a graph theoretic nontrivial (maximal) strong component C' of G' , and because there are no vertices (q_1, q_2) in G' with $\kappa_1(q_1) < k_1$ or $\kappa_2(q_2) < k_2$, we must have $\min(\kappa_i(\pi_i(C'))) = k_i$ for $i = 1, 2$. Thus, the algorithm will find at least one such graph theoretic strong component C' of G' and return “yes” and a correct witness.

The running time of the **Colors** procedure, exclusive of a call to the **Witness** procedure, is linear in the size of \mathcal{M} , that is, $O(|\Sigma| \cdot |Q_1| \cdot |Q_2|)$. Because the number of states of \mathcal{M} is bounded by $|Q_1| \cdot |Q_2|$, this is also an upper bound on the size of any nontrivial graph theoretic strong component of $G(\mathcal{M})$, so the length of any witness $u(v)^\omega$ returned is bounded by $O((|Q_1| \cdot |Q_2|)^2)$ and the overall running time is bounded by this quantity plus $O(|\Sigma| \cdot |Q_1| \cdot |Q_2|)$. \square

2.3 Inclusion and equivalence algorithms for DPAs

The inclusion problem for DPAs \mathcal{P}_1 and \mathcal{P}_2 over the same alphabet can be solved by looping over all odd k_1 in the range of κ_1 and all even k_2 in the range of κ_2 , calling the **Colors** procedure with inputs \mathcal{P}_1 , \mathcal{P}_2 , k_1 , and k_2 . If the **Colors** procedure returns any witness $u(v)^\omega$, then $u(v)^\omega \in \llbracket \mathcal{P}_1 \rrbracket \setminus \llbracket \mathcal{P}_2 \rrbracket$, and $u(v)^\omega$ is returned as a witness of non-inclusion. Otherwise, by Theorem 6, there is no ω -word w accepted by \mathcal{P}_1 and not accepted by \mathcal{P}_2 , and the answer “yes” is returned for the inclusion problem. Note that for $i = 1, 2$, the range of κ_i has at most $|Q_i|$ distinct elements.

Corollary 1. *There are algorithms for the inclusion and equivalence problems for two DPAs \mathcal{P}_1 and \mathcal{P}_2 over the same alphabet Σ that run in time bounded by $O((|Q_1| \cdot |Q_2|)^3 + |\Sigma|(|Q_1| \cdot |Q_2|)^2)$, where Q_i is the set of states of \mathcal{P}_i for $i = 1, 2$. A returned witness $u(v)^\omega$ has length $O((|Q_1| \cdot |Q_2|)^2)$.*

From Claims 3 and 4, and the fact that two colors suffice in the transformation of a DBA (or DCA) to a DPA, we have the following.

Corollary 2. *There are algorithms for the inclusion and equivalence problems for two DBAs (or DCAs) \mathcal{B}_1 and \mathcal{B}_2 over the same alphabet Σ that run in time bounded by $O((|Q_1| \cdot |Q_2|)^2 + |\Sigma| \cdot |Q_1| \cdot |Q_2|)$, where Q_i is the set of states of \mathcal{B}_i for $i = 1, 2$. A returned witness $u(v)^\omega$ has length $O((|Q_1| \cdot |Q_2|)^2)$.*

3 Inclusion and equivalence algorithms for DMAs

In this section we develop a polynomial time algorithm to solve the inclusion problem for two DMAs over the same alphabet. The proof proceeds in two parts: (1) a polynomial time reduction of the inclusion problem for two DMAs to the inclusion problem for a DBA and a DMA, and (2) a polynomial time algorithm for the inclusion problem for a DBA and a DMA.

3.1 Reduction of DMA inclusion to DBA/DMA inclusion

We first reduce the problem of inclusion for two arbitrary DMAs to the inclusion problem for two DMAs where the first one has just a single final state set. For $i = 1, 2$, define the DMA $\mathcal{U}_i = \langle Q_i, \Sigma, (q_i)_i, \delta_i, \mathcal{F}_i \rangle$, where \mathcal{F}_i is the set of final state sets for \mathcal{U}_i . Let the elements of \mathcal{F}_1 be $\{F_1, \dots, F_k\}$, and for each $j \in [1..k]$, let

$$\mathcal{U}_{1,j} = \langle Q_1, \Sigma, (q_1)_1, \delta_1, \{F_j\} \rangle,$$

that is, $\mathcal{U}_{1,j}$ is \mathcal{U}_1 with F_j as its only final state set. Then by the definition of DMA acceptance,

$$\llbracket \mathcal{U}_1 \rrbracket = \bigcup_{j=1}^k \llbracket \mathcal{U}_{1,j} \rrbracket,$$

which implies that to test whether $\llbracket \mathcal{U}_1 \rrbracket \subseteq \llbracket \mathcal{U}_2 \rrbracket$, it suffices to test for all $j \in [1..k]$ that $\llbracket \mathcal{U}_{1,j} \rrbracket \subseteq \llbracket \mathcal{U}_2 \rrbracket$.

Claim 7. *Suppose A is a procedure that solves the inclusion problem for two DMAs over the same alphabet, assuming that the first DMA has a single final state set. Then there is an algorithm that solves the inclusion problem for two arbitrary DMAs over the same alphabet, say \mathcal{U}_1 and \mathcal{U}_2 , which simply makes $|\mathcal{F}_1|$ calls to A , where \mathcal{F}_1 is the family of final state sets of \mathcal{U}_1 .*

Next we describe a procedure **SCC-to-DBA** that takes as inputs an automaton \mathcal{M} , a SCC F of \mathcal{M} , and a state $q \in F$, and returns a DBA $B(\mathcal{M}, F, q)$ that accepts exactly $L(\mathcal{M}, F, q)$, where $L(\mathcal{M}, F, q)$ is the set of ω -words w that visit only the states of F when processed by \mathcal{M} starting at state q , and visits each of them infinitely many times.

Assume the states in F are $\{q_0, q_1, \dots, q_{m-1}\}$, where $q_0 = q$. The DBA $B(\mathcal{M}, F, q)$ is $\langle Q', \Sigma, q_0, \delta', \{q_0\} \rangle$, where we define Q' and δ' as follows. We create new states $r_{i,j}$ for $i, j \in [0..m-1]$ such that $i \neq j$, and denote the set of these by R . We also create a new dead state d_0 . Then the set of states Q' is $Q \cup R \cup \{d_0\}$.

For δ' , the dead state d_0 behaves as expected: for all $\sigma \in \Sigma$, $\delta'(d_0, \sigma) = d_0$. For the other states in Q' , let $\sigma \in \Sigma$ and $i \in [0..m-1]$. If $\delta(q_i, \sigma)$ is not in F , then in order to deal with runs that would visit states outside of F , we define $\delta'(q_i, \sigma) = d_0$ and, for all $j \neq i$, $\delta'(r_{i,j}, \sigma) = d_0$.

Otherwise, for some $k \in [0..m-1]$ we have $q_k = \delta(q_i, \sigma)$. If $k = (i+1) \bmod m$, then we define $\delta'(q_i, \sigma) = q_k$, and otherwise we define $\delta'(q_i, \sigma) = r_{k, (i+1) \bmod m}$.

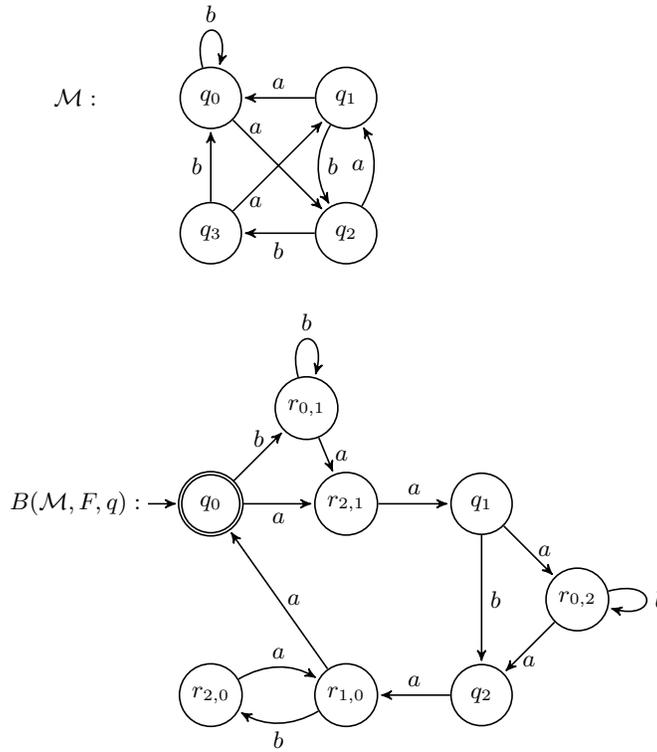


Fig. 1. Example of the construction of $B(\mathcal{M}, F, q)$ with $F = \{q_0, q_1, q_2\}$ and $q = q_0$.

For all $j \in [0..m-1]$ with $j \neq i$, if $k = j$, we define $\delta'(r_{i,j}, \sigma) = q_k$, and otherwise we define $\delta'(r_{i,j}, \sigma) = r_{k,j}$.

Intuitively, for an input from $L(\mathcal{M}, F, q)$, in $B(\mathcal{M}, F, q)$ the states q_i are visited in a repeating cyclic order: q_0, q_1, \dots, q_{m-1} , and the meaning of the state $r_{i,j}$ is that at this point in the input, \mathcal{M} would be in state q_i , and the machine $B(\mathcal{M}, F, q)$ is waiting for a transition that would arrive at state q_j in \mathcal{M} , in order to proceed to state q_j in $B(\mathcal{M}, F, q)$.⁴ An example of the construction is shown in Fig. 1; the dead state and unreachable states are omitted for clarity.

Lemma 2. *Let \mathcal{M} be an automaton with alphabet Σ and states Q , and let F a SCC of \mathcal{M} and $q \in F$. With these inputs, the procedure **SCC-to-DBA** returns the DBA $B(\mathcal{M}, F, q)$, which accepts the language $L(\mathcal{M}, F, q)$ and has $|F|^2 + 1$ states. The running time of **SCC-to-DBA** is $O(|\Sigma|(|Q| + |F|^2))$.*

⁴ This construction is reminiscent of the construction transforming a generalized Büchi into a Büchi automaton [7,4], by considering each state in F as a singleton set of a generalized Büchi, but here we need to send transitions to states outside F to a sink state.

Proof. Suppose w is in $L(\mathcal{M}, F, q)$. Let $q = s_0, s_1, s_2, \dots$ be the sequence of states in the run of \mathcal{M} from state q on input w . This run visits only states in F and visits each one of them infinitely many times. We next define a particular increasing sequence $i_{k,\ell}$ of indices in s , where k is a positive integer and $\ell \in [0, m-1]$. These indices mark particular visits to the states q_0, q_1, \dots, q_{m-1} in repeating cyclic order. The initial value is $i_{1,0} = 0$, marking the initial visit to q_0 . If $i_{k,\ell}$ has been defined and $\ell < m-1$, then $i_{k,\ell+1}$ is defined as the least natural number j such that $j > i_{k,\ell}$ and $s_j = q_{\ell+1}$, marking the next visit to $q_{\ell+1}$. If $\ell = m-1$, then $i_{k+1,0}$ is defined as the least natural number j such that $j > i_{k,\ell}$ and $s_j = q_0$, marking the next visit to q_0 .

There is a corresponding division of w into a concatenation of finite segments $w_{1,1}, w_{1,2}, \dots, w_{1,m-1}, w_{2,0}, \dots$ between consecutive elements in the increasing sequence of indices. An inductive argument shows that in $B(\mathcal{M}, F, q)$, the prefix of w up through $w_{k,\ell}$ arrives at the state q_ℓ , so that w visits q_0 infinitely often and is therefore accepted by $B(\mathcal{M}, F, q)$.

Conversely, suppose $B(\mathcal{M}, F, q)$ accepts the ω -word w . Let s_0, s_1, s_2, \dots be the run of $B(\mathcal{M}, F, q)$ on w , and let t_0, t_1, t_2, \dots be the run of \mathcal{M} starting from q on input w . An inductive argument shows that if $s_n = q_i$ then $t_n = q_i$, and if $s_n = r_{i,j}$ then $t_n = q_i$. Because the only way the run s_0, s_1, \dots can visit the final state q_0 infinitely often is to progress through the states q_0, q_1, \dots, q_{m-1} in repeating cyclic order, the run t_0, t_1, \dots must visit only states in F and visit each of them infinitely often, so $w \in L(\mathcal{M}, F, q)$.

The DBA $B(\mathcal{M}, F, q)$ has a dead state, and $|F|$ states for each element of F , for a total of $|F|^2 + 1$ states. The running time of **SCC-to-DBA** is linear in the size of \mathcal{M} and the size of the resulting DBA, that is, $O(|\Sigma|(|Q| + |F|^2))$, polynomial in the size of \mathcal{M} . \square

We now show that this construction may be used to reduce the inclusion of two DMAs to the inclusion of a DBA and a DMA. Recall that if \mathcal{A} is an acceptor and q is a state of \mathcal{A} , then \mathcal{A}^q denotes the acceptor \mathcal{A} with the initial state changed to q .

Lemma 3. *Let \mathcal{U}_1 be a DMA with automaton \mathcal{M}_1 and a single final state set F_1 . Let \mathcal{U}_2 be an arbitrary DMA over the same alphabet as \mathcal{U}_1 , with automaton \mathcal{M}_2 and family of final state sets \mathcal{F}_2 . Let \mathcal{M} denote the product automaton $\mathcal{M}_1 \times \mathcal{M}_2$ with unreachable states removed. Then $\llbracket \mathcal{U}_1 \rrbracket \subseteq \llbracket \mathcal{U}_2 \rrbracket$ iff for every state (q_1, q_2) of \mathcal{M} with $q_1 \in F_1$ we have $\llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket \subseteq \llbracket \mathcal{U}_2^{q_2} \rrbracket$.*

Proof. Suppose that for some state (q_1, q_2) of \mathcal{M} with $q_1 \in F_1$, we have $w \in \llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket \setminus \llbracket \mathcal{U}_2^{q_2} \rrbracket$. Let C_1 be the set of states visited infinitely often in $B(\mathcal{M}_1, F_1, q_1)$ on input w , and let C_2 be the set of states visited infinitely often in $\mathcal{U}_2^{q_2}$ on input w . Then $C_1 = F_1$ and $C_2 \notin \mathcal{F}_2$. Let u be a finite word such that $\mathcal{M}(u) = (q_1, q_2)$. Then $\text{Inf}_{\mathcal{M}_1}(uw) = C_1 = F_1$ and $\text{Inf}_{\mathcal{M}_2}(uw) = C_2$, so $uw \in \llbracket \mathcal{U}_1 \rrbracket \setminus \llbracket \mathcal{U}_2 \rrbracket$.

Conversely, suppose that $w \in \llbracket \mathcal{U}_1 \rrbracket \setminus \llbracket \mathcal{U}_2 \rrbracket$. For $i = 1, 2$ let $C_i = \text{Inf}_{\mathcal{M}_i}(w)$. Note that $C_1 = F_1$ and $C_2 \notin \mathcal{F}_2$. Let $w = xw'$, where x is a finite prefix of w that is sufficiently long that the run of \mathcal{M}_1 on w does not visit any state outside

C_1 after x has been processed, and for $i = 1, 2$ let $q_i = \mathcal{M}_i(x)$. Then (q_1, q_2) is a (reachable) state of \mathcal{M} , $q_1 \in F_1$, and the ω -word w' , when processed by \mathcal{M}_1 starting at state q_1 visits only states of $C_1 = F_1$ and visits each of them infinitely many times, that is, $w' \in \llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket$. Moreover, when w' is processed by \mathcal{M}_2 starting at state q_2 , the set of states visited infinitely often is C_2 , which is not in \mathcal{F}_2 . Thus, $w' \in \llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket \setminus \llbracket \mathcal{U}_2^{q_2} \rrbracket$. \square

To turn this into an algorithm to test inclusion for two DMAs, \mathcal{U}_1 with automaton \mathcal{M}_1 and a single final state set F_1 that is a SCC of \mathcal{M}_1 and \mathcal{U}_2 with automaton \mathcal{M}_2 , we proceed as follows. Construct the product automaton $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with unreachable states removed, and for each state (q_1, q_2) of \mathcal{M} , if $q_1 \in F_1$, construct the DBA $B(\mathcal{M}_1, F_1, q_1)$ and the DMA $\mathcal{U}_2^{q_2}$ and test the inclusion of language accepted by the DBA in the language accepted by the DMA. If all of these tests return “yes”, then the algorithm returns “yes” for the inclusion question for \mathcal{U}_1 and \mathcal{U}_2 . Otherwise, for the first test that returns “no” and a witness $u(v)^\omega$, the algorithm finds by breadth-first search a minimum length finite word u' such that $\mathcal{M}(u') = (q_1, q_2)$, and returns “no” and the witness $u'u(v)^\omega$.

Combining this with Claim 7, we have the following.

Theorem 8. *Let A be an algorithm to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet. There is an algorithm to test inclusion for an arbitrary pair of DMAs \mathcal{U}_1 and \mathcal{U}_2 over the same alphabet whose running time is linear in the sizes of \mathcal{U}_1 and \mathcal{U}_2 plus the time for at most $k \cdot |Q_1| \cdot |Q_2|$ calls to the procedure A , where k is the number of final state sets in \mathcal{U}_1 , and Q_i is the state set of \mathcal{U}_i for $i = 1, 2$.*

3.2 A DBA/DMA inclusion algorithm

In this section, we give a polynomial time algorithm **Incl-DBA-DMA** to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet. Assume the algorithm has inputs consisting of a DBA \mathcal{B} and a DMA \mathcal{U} , where

$$\mathcal{B} = \langle \Sigma, Q_1, (q_i)_1, \delta_1, F \rangle$$

and

$$\mathcal{U} = \langle \Sigma, Q_2, (q_i)_2, \delta_2, \mathcal{F} \rangle.$$

Let \mathcal{M}_1 denote the automaton of \mathcal{B} and \mathcal{M}_2 denote the automaton of \mathcal{U} . The **Incl-DBA-DMA** algorithm computes the product automaton $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with unreachable states removed. Note that any elements of \mathcal{F} that are not SCCs of \mathcal{M}_2 may be removed (in time linear in the size of \mathcal{U}) without affecting the language accepted by \mathcal{U} .

The overall strategy of the **Incl-DBA-DMA** algorithm is to seek a nontrivial graph theoretic strongly connected subset C of states of $G(\mathcal{M})$ such that $\pi_1(C) \cap F \neq \emptyset$ and $\pi_2(C) \notin \mathcal{F}$. If such a C is found, the algorithm calls the **Witness** procedure on inputs \mathcal{M} and C to find a witness $u(v)^\omega$ such that $\text{Inf}_{\mathcal{M}}(u(v)^\omega) =$

C . Because $\text{Inf}_{\mathcal{M}_1}(u(v)^\omega) = \pi_1(C)$ and $\pi_1(C) \cap F \neq \emptyset$, $u(v)^\omega$ is accepted by \mathcal{B} . Because $\text{Inf}_{\mathcal{M}_2}(u(v)^\omega) = \pi_2(C)$ and $\pi_2(C) \notin \mathcal{F}$, $u(v)^\omega$ is not accepted by \mathcal{U} .

Once the product automaton \mathcal{M} has been computed, the **Incl-DBA-DMA** algorithm proceeds as follows.

Step one. Compute the graph theoretic nontrivial strongly connected components of $G(\mathcal{M})$, say C_1, C_2, \dots, C_k . If for any i with $i \in [1..k]$ we have $\pi_1(C_i) \cap F \neq \emptyset$ and $\pi_2(C_i) \notin \mathcal{F}$, return “no” and the witness returned by the **Witness** procedure on inputs \mathcal{M} and C_i .

Step two. Otherwise, for each $i \in [1..k]$ such that $C_i \cap F \neq \emptyset$, process C_i as follows. For each element $F_j \in \mathcal{F}$ such that $F_j \subseteq \pi_2(C_i)$, and for each state $q \in F_j$, compute the graph theoretic nontrivial strongly connected components, say D_1, D_2, \dots, D_m , of the subgraph of $G(\mathcal{M})$ induced by all vertices (q_1, q_2) such that $q_1 \in \pi_1(C_i)$ and $q_2 \in F_j \setminus \{q\}$. For each such D_s , test whether $\pi_1(D_s) \cap F \neq \emptyset$ and $\pi_2(D_s) \notin \mathcal{F}$. If so, return “no” and the witness returned by **Witness** on inputs \mathcal{M} and D_s .

Step three. If none of the tests in Steps one or two return “no” and a witness, then return “yes”.

Theorem 9. *The **Incl-DBA-DMA** algorithm solves the inclusion problem for an arbitrary DBA \mathcal{B} and an arbitrary DMA \mathcal{U} over the same alphabet Σ . Any returned witness $u(v)^\omega$ has length $O((|Q_1| \cdot |Q_2|)^2)$, and the running time is $O(|\Sigma| \cdot |Q_1| \cdot |Q_2| + |\mathcal{F}| \cdot |Q_1| \cdot |Q_2|^2)$, where Q_1 is the state set of \mathcal{B} , Q_2 is the state set of \mathcal{U} , and \mathcal{F} is the family of final state sets of \mathcal{U} .*

Proof. To establish the correctness of the **Incl-DBA-DMA** algorithm, we argue as follows. Suppose the returned value is “no” with a witness $u(v)^\omega$. Then the algorithm must have found a graph theoretic nontrivial strongly connected component C of $G(\mathcal{M})$ with $\pi_1(C) \cap F \neq \emptyset$ and $\pi_2(C) \notin \mathcal{F}$ and called the **Witness** procedure with inputs \mathcal{M} and C , which returned the witness $u(v)^\omega$, which correctly witnesses the answer “no”. Thus, in this case, the returned value is correct and the witness has length at most $O((|Q_1| \cdot |Q_2|)^2)$.

Suppose for the sake of contradiction that the algorithm **Incl-DBA-DMA** returns “yes” but should not, that is, there exists an ω -word w such that $w \in \llbracket \mathcal{B} \rrbracket$ and $w \notin \llbracket \mathcal{U} \rrbracket$. Let C denote $\text{Inf}_{\mathcal{M}}(w)$, the set of states visited infinitely often in the run of \mathcal{M} on input w . Then because $w \in \llbracket \mathcal{B} \rrbracket$, $\pi_1(C) \cap F \neq \emptyset$. And because $w \notin \llbracket \mathcal{U} \rrbracket$, $\pi_2(C) \notin \mathcal{F}$.

Clearly, C is a subset of a unique C_i computed in Step one, and $\pi_1(C_i) \cap F \neq \emptyset$. It must be that $\pi_2(C_i) \in \mathcal{F}$, because otherwise the algorithm would have returned “no” with the witness computed from C_i . Let F_1, F_2, \dots, F_ℓ denote the elements of \mathcal{F} that are subsets of $\pi_2(C_i)$.

Consider the collection

$$S = \{F_r \mid r \in [1..\ell] \wedge \pi_2(C) \subseteq F_r\},$$

of all the F_r contained in $\pi_2(C_i)$ that contain $\pi_2(C)$. The collection S is nonempty because $C \subseteq C_i$, and therefore $\pi_2(C) \subseteq \pi_2(C_i)$, and $\pi_2(C_i) \in \mathcal{F}$, so at least $\pi_2(C_i)$ is in S . Let F_j denote a minimal element (in the subset ordering) of S .

Then $\pi_2(C) \subseteq F_j$ but because $\pi_2(C) \notin \mathcal{F}$, it must be that $\pi_2(C) \neq F_j$. Thus, there exists some $q \in F_j$ that is not in $\pi_2(C)$. When the algorithm considers this F_j and q , then because $\pi_2(C) \subseteq F_j \setminus \{q\}$, C is contained in one of the graph theoretic nontrivial strongly connected components D_s of the subgraph of $G(\mathcal{M})$ induced by the vertices (q_1, q_2) such that $q_1 \in C_i$ and $q_2 \in F_j \setminus \{q\}$.

Because $C \subseteq D_s$, and $\pi_1(C) \cap F \neq \emptyset$, we have $\pi_1(D_s) \cap F \neq \emptyset$. Also, $\pi_2(C) \subseteq \pi_2(D_s) \subseteq F_j$, but because $q \notin \pi_2(D_s)$, $\pi_2(D_s)$ is a proper subset of F_j . When the algorithm considers D_s , because $\pi_1(D_s) \cap F \neq \emptyset$, it must find that $\pi_2(D_s) \in \mathcal{F}$, or else it would have returned “no”. But then $\pi_2(D_s)$ is in S and is a proper subset of F_j , contradicting our choice of F_j as a minimal element of S . Thus, if all the tests in Steps one and two pass, the returned value “yes” is correct.

To analyze the running time of the **Incl-DBA-DMA** algorithm, consider first the task of determining for a graph theoretic nontrivial strongly connected component C of a subgraph of $G(\mathcal{M})$, whether $\pi_1(C) \cap F \neq \emptyset$ and whether $\pi_2(C) \in \mathcal{F}$. We assume that time linear in $|C|$ suffices for these tests, using hash tables constructed in time linear in the sizes of \mathcal{B} and \mathcal{U} .

In Step one, the computation of the graph theoretic strongly connected components may be carried out in time linear in the size of \mathcal{M} . Checking each resulting component C_i can be done in time linear in $|C_i|$, so the total time is linear in the size of \mathcal{M} , that is $O(|\Sigma| \cdot |Q_1| \cdot |Q_2|)$.

For Step two, for each component C_i such that $\pi_1(C_i) \cap F \neq \emptyset$ there are at most $|\mathcal{F}|$ sets F_s to consider, and for each of them at most $|F_s|$ computations of nontrivial graph theoretic strongly connected components of subgraphs of $G(\mathcal{M})$, and tests of each of the resulting components. Each F_s has size at most $|Q_2|$, so the subgraph of $G(\mathcal{M})$ considered has size at most $|C_i| \cdot |Q_2|$, and the total time in Step two is $O(|\mathcal{F}| \cdot |Q_1| \cdot |Q_2|^2)$. \square

Combining Theorem 8, Theorem 9, and the reduction of equivalence to inclusion, we have the following.

Corollary 3. *There are polynomial time algorithms to solve the inclusion and equivalence problems for two arbitrary DMAs \mathcal{U}_1 and \mathcal{U}_2 over the same alphabet Σ . If for $i = 1, 2$, Q_i is the set of states and \mathcal{F}_i is the family of final state sets of \mathcal{U}_i , then the length of any returned witness is $O(|Q_1| \cdot |Q_2|^2)$ and the total running time is*

$$O(|\mathcal{F}_1| \cdot |\mathcal{F}_2| \cdot |Q_1|^2 \cdot |Q_2|^3 + |\Sigma| \cdot |\mathcal{F}_1| \cdot |Q_1|^2 \cdot |Q_2|^2).$$

The running time bound reflects repeating once for each element of \mathcal{F}_1 and each pair (q_1, q_2) in $Q_1 \times Q_2$, the cost of a call to the procedure to test inclusion for a DBA and a DMA.

4 Computing the right congruence automaton

Let \mathcal{A} be a DBA, DCA, DPA, or DMA. Recall that \mathcal{A}^q is the acceptor \mathcal{A} with the initial state changed to q . Then $\llbracket \mathcal{A}^q \rrbracket$ is the set of all ω -words accepted from the state q . Thus, if q_1 and q_2 are two states of \mathcal{A} , testing the equivalence of \mathcal{A}^{q_1} to \mathcal{A}^{q_2} determines whether these two states have the same right congruence class, and, if not, returns a witness $u(v)^\omega$ that is accepted from exactly one of the two states. The following is a consequence of Corollaries 1, 2, and 3.

Lemma 4. *There is a polynomial time procedure to test whether two states of an arbitrary DBA, DCA, DPA, or DMA \mathcal{A} have the same right congruence class, returning the answer “yes” if they do, and returning “no” and a witness $u(v)^\omega$ accepted from exactly one of the states if they do not. A returned witness $u(v)^\omega$ has length $O(|Q|^4)$, where Q is the set of states of \mathcal{A} .*

This in turn can be used in an algorithm **Right-Con** to construct the right congruence automaton for a given DBA (or DCA, DPA, or DMA). Let the input acceptor be

$$\mathcal{A} = \langle \Sigma, Q, q_i, \delta, \alpha \rangle.$$

The algorithm constructs an automaton

$$\mathcal{M} = \langle \Sigma, Q', \varepsilon, \delta' \rangle,$$

isomorphic to the right congruence automaton of $\llbracket \mathcal{A} \rrbracket$ in which the states are represented by finite words and the initial state is the empty word ε .

We describe the process of constructing Q' and δ' , where Q' initially contains just the empty word, and δ' is completely undefined. While there exists a word $x \in Q'$ and a symbol $\sigma \in \Sigma$ such that $\delta'(x, \sigma)$ has not yet been defined, loop through the words $y \in Q'$ and ask whether the states $\delta(q_i, x\sigma)$ and $\delta(q_i, y)$ have the same right congruence class in \mathcal{A} . If so, then define $\delta'(x, \sigma)$ to be y . If no such y is found, then the finite word $x\sigma$ is added as a new state to Q' , and the process continues.

This process must terminate because the right congruence automaton of \mathcal{A} cannot have more than $|Q|$ states. When it terminates, the automaton \mathcal{M} is isomorphic to the right congruence automaton of \mathcal{A} .

Note that each time the equivalence algorithm returns “no” to a call with $\delta(q_i, x\sigma)$ and $\delta(q_i, y)$, it also returns a witness $u(v)^\omega$ such that exactly one of $x\sigma u(v)^\omega$ and $y u(v)^\omega$ is accepted by \mathcal{A} . Thus, if the **Right-con** algorithm collects, for each new state $x\sigma$ added to Q' , the set of witnesses $u(v)^\omega$ distinguishing it from the previous elements of Q' , the resulting set D of witnesses are sufficient to distinguish every pair of states of the final automaton \mathcal{M} .

Theorem 10. *The **Right-con** algorithm with input \mathcal{A} (a DBA, DCA, DPA or DMA) runs in polynomial time and returns \mathcal{M} , an automaton isomorphic to the right congruence automaton of $\llbracket \mathcal{A} \rrbracket$, and D , a set of witnesses $u(v)^\omega$ such that for any $x_1, x_2 \in \Sigma^*$, if $\mathcal{M}(x_1) \neq \mathcal{M}(x_2)$ then there exists some $u(v)^\omega \in D$ such that exactly one of $x_1 u(v)^\omega$ and $x_2 u(v)^\omega$ is in $\llbracket \mathcal{A} \rrbracket$.*

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