p-ADIC L-FUNCTIONS FOR $\mathrm{GL}_2 \times \mathrm{GU}(1)$

by

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Abstract. — Let F be a totally real field and let E/F be a CM quadratic extension. We construct a p-adic L-function attached to Hida families for the group $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2 \times \mathrm{Res}_{E/\mathbf{Q}}\mathrm{GL}_1$. It interpolates critical Rankin–Selberg L-values at all classical points corresponding to representations $\pi \boxtimes \chi$ with the weights of χ smaller than the weights of π. This confirms a conjecture of the author, making related results on the p-adic Beĭlinson conjecture and its Iwasawa-theoretic variants unconditional.

Our p-adic L-function agrees with a previous result of Hida when E/F splits above p, and it is new otherwise. We build it as a ratio of families of global and local Waldspurger zeta integrals, the latter constructed using the local Langlands correspondence in families.

In an appendix of possibly independent recreational interest, we give a reality-TV-inspired proof of an identity concerning double factorials.

Contents

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1. Introduction

Let F be a totally real field and E/F a CM quadratic extension, and let p be a rational prime. In [Dis/b], we prove, first, the p-adic Beilinson–Bloch–Kato conjecture in analytic rank 1 for p-ordinary, selfdual motives attached to Hilbert modular forms for F, or their twists by Hecke characters of E of lower weights; and secondly, one divisibility in an Iwasawa Main Conjecture for cyclotomic derivatives of p-adic L-functions. Both results, which are new or partly new even when $F = \mathbf{Q}$ and E/F splits at p (and have some new applications), depend in an essential way on having a p-adic L-function $\mathscr{L}_p(\mathscr{V})$ with 'maximally' general and precise interpolation properties. The construction of $\mathscr{L}_p(\mathscr{V})$ is the main result of this paper. It is new at least when E/F does not split above p; for a discussion of previous related works see § 1.1.6.

As for further arithmetic directions,(1) the main remaining goal is perhaps the full Iwasawa Main Conjecture for $\mathscr{L}_p(\mathscr{V})$. This was proved by Skinner–Urban [SU14] and Wan [Wan15] in the split case. In the non-split case, results toward it (when $F = \mathbf{Q}$) were recently obtained by Büyükboduk–Lei [BL]. A second goal is the remaining divisibility in the Main Conjecture for the cyclotomic derivative of $\mathscr{L}_p(\mathscr{V})$ (cf. [Dis/b, Theorem E]); in view of the p-adic Gross–Zagier formula of [Dis/b], this is equivalent to a suitable generalisation of Perrin-Riou's main conjecture for Heegner points, which in its original form was recently proved by Burungale–Castella–Kim [BCK].

Besides the specifics of our context, this work may also be viewed as a case study in:

- p-adic L-functions in universal families which exactly interpolate ratios of L-values;(2)
- constructing p-adic L-functions as ratios of families of global and local zeta integrals, where the latter are interpolated using the local Langlands Correspondence in families (see [Dis20] and references therein). As far as the author knows, the present work is the first instance where this method is used for a non-abelian family.

1.1. Statement of the main result. — We move toward stating our main theorem, leaving a few of of the detailed definitions of the objects involved to the body of the paper.

1.1.1. *p-adic automorphic representations.* — Let

$$\mathrm{G} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2, \qquad \mathrm{H} := \mathrm{Res}_{E/\mathbf{Q}}\mathrm{GL}_1.$$

If v_0 is a place of $\mathbf{Q}$, we denote by Σ_{v_0} the set of embeddings $F_{v_0} \hookrightarrow \overline{\mathbf{Q}}_{v_0}$. A (numerical) v_0-adic weight for G is a tuple $\underline{w} := (w_0, w = (w_\tau)_{\tau\in\Sigma_{v_0}})$ of integers, all of the same parity, such that $w_\tau \geq 0$ for all τ. It is said cohomological if $w_\tau \geq 2$ for all τ. A weight for H is a tuple $\underline{l} = (l_0, l = (l_\tau))$ of integers of the same parity. Finally, if $\underline{w}$ and $\underline{l}$ are weights for G and H, the associated *contracted weight* for $\mathrm{G}\times\mathrm{H}$ is(3)

$$(w_0 + l_0, w, l).$$

If $\underline{w}$ is a p-adic weight (say, for G) and $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ is an embedding, we denote $\underline{w}^\iota := (w_0, (w_\tau)_{\iota\circ\tau\colon F\hookrightarrow\mathbf{C}})$. (In fact $\underline{w}^\iota$ only depends on $\iota_{|L}$ if $\underline{w}$ is rational over the finite extension L of $\mathbf{Q}_p$ in the sense that $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$ fixes $\underline{w}$.) An automorphic representation of archimedean weight $\underline{w}$ is a complex automorphic representation π of $\mathrm{G}(\mathbf{A})$ such that $\pi_\infty = \pi_{\underline{w}} := \otimes_{\tau\colon F\hookrightarrow\mathbf{R}}\pi_{(w_0,w_\tau)}$, where $\pi_{(w_0,w_\tau)}$ is the discrete series of $\mathrm{GL}_2(F_\tau)$ of weight w_τ and central character $z\mapsto z^{w_0}$. If L is p-adic, we define an *automorphic representation of* $\mathrm{G}(\mathbf{A})$ *of weight* $\underline{w}$ *over* L to be a representation π of

(1) This discussion has no ambition of being either a comprehensive research programme or a comprehensive survey of the growing literature on the subject. Moreover, we have entirely left out both the case of non-ordinary families, and the p-adic L-function with complementary (to (1.1.5)) interpolation range introduced in [BDP13].

(2) This can be considered largely a matter of emphasis, since it is not difficult to deduce a few results of this type from Hida's work, cf. [Hid96].

(3) A contracted weight is the same as a weight for $(\mathrm{G}\times\mathrm{H})^1 := \{(g,h) : \det(g) = N_{E/F}(h)\} \subset \mathrm{G}\times\mathrm{H}$. This is in fact the true group governing our constructions.

$\mathrm{G}(\mathbf{A}^\infty)$ on an L-vector space, such that for every $\iota\colon L \hookrightarrow \mathbf{C}$, the representation

$$\pi^\iota := \pi \otimes_{L,\iota} \pi_{\infty,\underline{w}^\iota}$$

of $\mathrm{G}(\mathbf{A})$ is automorphic.(4)

To the representation π over L is attached a 2-dimensional representation V_π of $G_F := \mathrm{Gal}(\overline{F}/F)$ characterised by $L(s, V_{\pi,v}) = L(s - 1/2, \pi_v)$ (this is the 'Hecke' normalisation of the Langlands correspondence, cf. [Del73, §3.2]). We say that π is *ordinary* if for each $v|p$, the restriction $V_{\pi,v}$ of V_π to a decomposition group G_{F_v} at v admits (possibly after taking a finite extension L'/L) a nontrivial filtration

$$0 \to V_{\pi,v}^+ \to V_{\pi,v} \to V_{\pi,v}^- \to 0 \tag{1.1.1}$$

such that the character $\alpha_{\pi,v}^\circ\colon F_v^\times \to L'^\times$ corresponding to(5) $V_{\pi,v}^+(-1)$ has values in $\mathscr{O}_{L'}^\times$.

Let L be a p-adic field splitting E, suppose chosen for each $\tau\colon F \hookrightarrow L$ an extension τ' to E,(6) and let $\tau'^c = \tau' \circ c$ for the complex conjugation c of E/F. A Hecke character of H of weight $\underline{l}$ over the p-adic field L is a locally algebraic character $\chi\colon E^\times\backslash\mathbf{A}_E^{\infty,\times} \to L^\times$ such that

$$\chi(t_p) = \prod_{\tau\colon F\hookrightarrow L} \tau'(t_p)^{(l_\tau + l_0)/2)}\tau'^c(t_p)^{(-l_\tau+l_0)/2)}$$

for all t_p in some neighbourhood of $1 \in E_p^\times$. We let V_χ be the 1-dimensional G_E-representation corresponding to χ.

1.1.2. *L-values.* — Let π, respectively χ, be a complex automorphic representation of $\mathrm{G}(\mathbf{A})$, respectively $\mathrm{H}(\mathbf{A})$, and let π_E denote the base-change of π to E. Let V_v (respectively $V_{\pi,v}$) be the restriction to $G_{E_v} := \prod_{w|v} G_{E,w}$ (respectively G_{F_v}) of the Galois representation associated with $\pi_E \otimes \chi$ (respectively π) if v is finite, and the Hodge structure associated with $\pi_{E,v} \otimes \chi_v$ (respectively π_v) if $v|\infty$. Let us also introduce the convenient notation

$$\text{“}V_{(\pi,\chi),v} := (V_{\pi,v} \otimes \mathrm{Ind}_{F_v}^{E_v}\chi_v) \ominus \mathrm{ad}(V_\pi)(1)\text{”}$$

(to be thought of as referring to a 'virtual motive').

Let $\eta\colon F_{\mathbf{A}}^\times/F^\times \to \{\pm 1\}$ be the character associated with E/F, and let

$$\begin{aligned}\mathscr{L}(V_{(\pi,\chi),v}, 0) &:= \frac{\zeta_{F,v}(2)L(1/2, \pi_{E,v}\otimes\chi_v)}{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})} \cdot \begin{cases} 1 & \text{if } v\nmid\infty \\ \pi^{-1} & \text{if } v|\infty \end{cases} \in \mathbf{C}, \\ \mathscr{L}(V_{(\pi,\chi)}, 0) &:= \prod_v \mathscr{L}(V_{(\pi,\chi),v}, 0),\end{aligned} \tag{1.1.2}$$

where the product (in the sense of analytic continuation) is over all places. These are the L-values we will interpolate.

1.1.3. *Interpolation factors.* — Let L be a finite extension of $\mathbf{Q}_p$, let π be an ordinary automorphic representations of $\mathrm{G}(\mathbf{A})$ over L, with a locally algebraic central character $\omega_\pi\colon \mathbf{A}_F^{\infty,\times} \to L^\times$, and let $\chi\colon \mathrm{H}(F)\backslash\mathrm{H}(\mathbf{A}) \to L^\times$ be a locally algebraic character, with $\omega_\chi := \chi_{|\mathbf{A}_F^{\infty,\times}}$. Let $\iota\colon L \hookrightarrow \mathbf{C}$ be an embedding and let $\psi = \prod_v \psi_v\colon F\backslash\mathbf{A}_F \to \mathbf{C}^\times$ be the standard additive character such that $\psi_\infty(x) = e^{2\pi i \mathrm{Tr}_{F_\infty/\mathbf{R}}(x)}$.

If $v|p$, let $\mathrm{ad}(V_{\pi,v})(1)^{++} := \mathrm{Hom}\,(V_{\pi,v}^-, V_{\pi,v}^+)$, and define

$$e_v(V_{(\pi^\iota,\chi^\iota)}) = \frac{\prod_{w|v}\gamma(\iota\mathrm{WD}(V^+_{\pi,v|G_{E,w}} \otimes V_{\chi,w}), \psi_{E,w})^{-1}}{\gamma(\iota\mathrm{WD}(\mathrm{ad}(V_{\pi,v})(1)^{++}, \psi_v)^{-1}} \cdot \mathscr{L}(V_{(\pi^\iota,\chi^\iota),v})^{-1},$$

(4)This definition is slightly different from, but equivalent to, the one adopted in [Dis/b], whose flexibility won't be needed here.
(5)The class field theory map $F_v^\times \to G_{F_v}^{\mathrm{ab}}$ is normalised by sending uniformisers to geometric Frobenii.
(6)This will only intervene in the numerical labelling of the weights.

where ιWD is the functor from potentially semistable Galois representations to complex Weil–Deligne representations of [Fon94], the inverse Deligne–Langlands γ-factor is $\gamma(W,\psi_v)^{-1} = L(W)/\varepsilon(W,\psi_v)L(W^*(1))$,(7) and $\psi_{E,w} = \psi_v \circ \mathrm{Tr}_{E_w/F_v}$.

Let $k_0 \in \mathbf{Z}$ be such that the archimedean component of $\omega = \omega_\pi\omega_\chi$ is $\omega_\infty(x) = x^{k_0}$. We define

$$e_\infty(V_{(\pi^\iota,\chi^\iota)}) = i^{k_0[F:\mathbf{Q}]},$$

and

$$e_{p\infty}(V_{(\pi^\iota,\chi^\iota)}) := e_\infty(V_{(\pi^\iota,\chi^\iota)}) \cdot \prod_{v|p} e_v(V_{(\pi^\iota,\chi^\iota)}) \tag{1.1.3}$$

1.1.4. *Hida families.* — Let $U^p_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be any open compact subgroup, and let $\mathbf{T}^{\mathrm{sph,ord}}_{U^p}$ be the p-(nearly) ordinary spherical Hecke algebra acting on ordinary p-adic modular cuspfoms for G of tame level U^p_{G}.

A *cuspidal Hida family* $\mathscr{X}_{\mathrm{G}}$ is an irreducible component of the space $\mathscr{Y}_{\mathrm{G},U^p_{\mathrm{G}}} := \mathrm{Spec}\,\mathbf{T}^{\mathrm{sph,ord}}_{U^p_{\mathrm{G}},\mathbf{Q}_p}$ for some U^p_{G}. It is a scheme finite flat over $\mathrm{Spec}\,\mathbf{Z}_p[\![T_1,\ldots,T_{[F:\mathbf{Q}]+1+\delta_{F,p}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (where $\delta_{F,p}$ is the p-Leopoldt defect of F), coming with a dense ind-finite subscheme

$$\mathscr{X}^{\mathrm{cl}}_{\mathrm{G}} \subset \mathscr{X}_{\mathrm{G}}$$

of *classical points*, and a locally free sheaf $\mathscr{V}_{\mathrm{G}}$ of rank 2 endowed with a representation of $G_F := \mathrm{Gal}(\overline{F}/F)$. To each $x \in \mathscr{X}^{\mathrm{cl}}$ is associated an automorphic representation π_x of $\mathrm{G}(\mathbf{A}^\infty)$ over $\mathbf{Q}_p(x)$, and the fibre $\mathscr{V}_{\mathrm{G}|x} \cong V_{\pi_x}$. The (numerical) weight of x is defined to be the weight of π_x.

Let $U^p_{\mathrm{H}} \subset \mathrm{H}(\mathbf{A}^{p\infty})$ be an open compact subgroup. We define

$$\mathscr{Y}_{\mathrm{H}} = \mathscr{Y}_{\mathrm{H},U^p_{\mathrm{H}}} := \mathrm{Spec}\,\mathbf{Z}_p[\![\mathrm{H}(\mathbf{Q})\backslash\mathrm{H}(\mathbf{A}^{p\infty})/U^p]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \tag{1.1.4}$$

where the topology on $\mathrm{H}(\mathbf{Q})\backslash\mathrm{H}(\mathbf{A}^{p\infty})/U^p$ is profinite; it comes with a universal character $\chi_{\mathrm{univ}}\colon \mathrm{H}(\mathbf{Q})\backslash\mathrm{H}(\mathbf{A}^\infty) \to \mathscr{O}(\mathscr{Y}_{\mathrm{H},U_{\mathrm{H}}})^\times$, identified with a G_E-representation $\mathscr{V}_{\mathrm{H}}$ of rank 1, and a dense ind-finite subscheme $\mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}} \subset \mathscr{Y}_{\mathrm{H}}$, whose points y correspond to U^p_{H}-invariant locally algebraic Hecke characters χ_y of H over $\mathbf{Q}_p(y)$. The weight of y is defined to be the weight of χ_y.

Finally, the ordinary eigenvariety for $\mathrm{G}\times\mathrm{H}$ is

$$\mathscr{Y}_{\mathrm{G}\times\mathrm{H}} := \mathscr{Y}_{\mathrm{G}}\hat{\times}\mathscr{Y}_{\mathrm{G}} := \mathrm{Spec}\,\mathbf{T}^{\mathrm{sph,ord}}_{U^p_{\mathrm{G}},\mathbf{Z}_p}\hat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[\![\mathrm{H}(\mathbf{Q})\backslash\mathrm{H}(\mathbf{A}^{p\infty})/U^p]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

A Hida family for $\mathrm{G}\times\mathrm{H}$ is an irreducible component of $\mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$. We now isolate an interesting subspace of $\mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$. Let $\omega_{\mathrm{G}}\colon F^\times\backslash\mathbf{A}^{\infty,\times}_F \to \mathscr{O}(\mathscr{Y}_{\mathrm{G}})^\times$ be the character giving the action of the centre of $\mathrm{G}(\mathbf{A})$ on p-adic modular forms, let $\omega_{\mathrm{H}} := \chi_{\mathrm{univ}|\mathbf{A}^{\infty,\times}_{\mathrm{F}}}$, and let

$$\omega := \omega_{\mathrm{G}}\omega_{\mathrm{H}}\colon F^\times\backslash\mathbf{A}^{\infty,\times}_F \to \mathscr{O}(\mathscr{Y}_{\mathrm{G}\times\mathrm{H}})^\times.$$

The *self-dual locus*

$$\mathscr{Y}^{\mathrm{sd}}_{\mathrm{G}\times\mathrm{H}} \subset \mathscr{Y}^{\mathrm{sd}}_{\mathrm{G}\times\mathrm{H}}$$

is the closed subspace defined by $\omega = \mathbf{1}$.

1.1.5. *Main theorem.* — Let $\mathscr{X}$ be a Hida family for $\mathrm{G}\times\mathrm{H}$. Denote $\mathscr{X}^{\mathrm{sd}} := \mathscr{X} \cap \mathscr{Y}^{\mathrm{sd}}_{\mathrm{G}\times\mathrm{H}}$, and $\mathscr{X}^{\mathrm{cl,sd}} := \mathscr{X}^{\mathrm{cl}} \cap \mathscr{X}^{\mathrm{sd}}$.

Throughout this paper, if $\mathscr{X}$ is a scheme over a characteristic-zero field L, we identify a point $x \in \mathscr{X}(\mathbf{C})$ with a pair (x_0,ι) where $x_0 \in \mathscr{X}$ is a point and $\iota\colon L(x) \hookrightarrow \mathbf{C}$ is an embedding. (We will often abuse notation by writing x in place of x_0.)

Theorem A. — *Assume that $\mathscr{X}^{\mathrm{sd}}$ is non-empty. There exists a unique meromorphic function*

$$\mathscr{L}_p(\mathscr{V}) \in \mathscr{K}(\mathscr{X})$$

(7) The normalisations of L- and ε-factors are as in [Tat79].

whose polar locus $\mathscr{D}$ *does not intersect* $\mathscr{X}^{\mathrm{cl,sd}}$*, such that for each* $(x,y) \in \mathscr{X}^{\mathrm{cl}}(\mathbf{C}) - \mathscr{D}(\mathbf{C})$ *whose contracted weight* (k_0, w, l) *satisfies*

$$|l_\tau| \leq w_\tau - 2, \qquad |k_0| \leq w_\tau - |l_\tau| - 2 \qquad \text{for all } \tau \in \Sigma_\infty, \tag{1.1.5}$$

we have

$$\mathscr{L}_p(\mathscr{V})(x,y) = e_{p\infty}(V_{(\pi_x,\chi_y)}) \cdot \mathscr{L}(V_{(\pi_x,\chi_y)}, 0). \tag{1.1.6}$$

Here, if $(x_0, y_0) \in \mathscr{X}^{\mathrm{cl}}$ *is the point underlying* (x,y) *and* $\iota\colon \mathbf{Q}_p(x_0,y_0) \hookrightarrow \mathbf{C}$ *is the corresponding embedding, we have denoted* $\pi_x = \pi_{x_0}^\iota, \chi_y = \chi_{y_0}^\iota$*, and the interpolation factor is as in* (1.1.3)*.*

The value of the interpolation factor agrees with the general conjectures of Coates and Perrin-Riou (see [Coa91]).

Remark 1.1.1. — This confirms [Dis/b, Conjecture (L_p) of § 1.4], up to the possible poles. (The power of the discriminant in *loc. cit.*, as well as the precise description of the ring of rationality, are not correct.)

1.1.6. *Previous related work.* — When E/F splits above p, Theorem A may be essentially deduced from the main result of [Hid91] (see also [Hid09]), cf. [Dis/b, § 7.1]. Hida's method uses the Rankin–Selberg integral, whereas ours uses Waldspurger's variant [Wal85] based on the Weil representation (as discussed below).

The numerator of our L-value is a special case of the standard L-function for $\mathrm{GL}_2 \times \mathrm{GL}_1$ over E, and when so considered our p-adic L-function is a multiple of the restriction to some base-change locus of one constructed by Januszewski [Jan] (using the method of modular symbols); however that function does not have the canonical interpolation properties that we seek.

Finally, when $F = \mathbf{Q}$ and p is inert in E, a variant of $\mathscr{L}_p(\mathscr{V})$ was recently constructed by Loeffler and Büyükboduk–Lei (see [BL, §B.4]) under various local restrictions.

1.2. Idea of proof and organisation of the paper. — The proof combines the strategy of Hida [Hid91] with that of [Dis17, Proof of Theorem A], where we had constructed the 'slices' $\mathscr{L}_p(\mathscr{V})(x,-)$ for $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$ of weight 2.

We start from Waldspurger's [Wal85] integral representation of Rankin–Selberg type

$$(f, I(\phi,\chi)) = \mathscr{L}(V_{(\pi,\chi)}, 0) \cdot \prod_v R_v(W_v, \phi_v, \chi_v, \psi_v) \tag{1.2.1}$$

where (,) is a normalised Petersson product, f is a form in π with Whittaker function W, the form $I(\phi)$ is a mixed theta-Eisenstein series depending on a certain Schwartz function ϕ, and the R_v are normalised local integrals.

In § 2, we discuss the general setup. In § 3, we make a judicious choice of ϕ_v at the places $v|p\infty$ and interpolate the ordinary projection of $I(\phi)$ into a $\mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$-adic modular form. In § 4, we interpolate R_v for $v \nmid p\infty$ using the local Langlands correspondence in families; we compute R_v for $v|p\infty$, which yield the interpolation factors in (1.1.6); and finally, we use (1.2.1) to define $\mathscr{L}_p$ as a quotient of the global and local (away from $p\infty$) families of zeta integrals.

In Appendix A, we give a TV-inspired bijective proof of a combinatorial lemma occurring in § 3.

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2. p-adic modular forms and Hida families

The material of this section is largely due to Hida (see [Hid91, §§ 1–3, 7] and references therein).

2.1. Notation and preliminaries. — The notation introduced in the present subsection (or in the introduction) will be used throughout the paper unless otherwise noted.

2.1.1. *General notation.* — The following notational choices are largely standard.

- The fields F and E are as fixed in the introduction unless specified otherwise; if $*$ denotes a place of $\mathbf{Q}$ or a finite set thereof, we denote by S_* the set of places of F above $*$;
- we denote by D_F, D_E and $D_{E/F}$, respectively, the absolute discriminants of F and E and relative discriminant of E/F; for a finite place v of F we denote by $d_v \in F_v$ a generator of the different ideal of F and by $D_v \in F_v$ a generator of the relative discriminant ideal;
- we denote by $<$ the partial order on F given by $x < y$ if and only if $\tau(x) < \tau(y)$ for all $\tau \in \Sigma_\infty$; we denote $\mathbf{R}^+ := \{x \in \mathbf{R} \,|\, x > 0\}$ and $F^+ := \{x \in F \,|\, x > 0\} \subset F^\times$;
- $\mathbf{A}$ is the ring of adèles of $\mathbf{Q}$; if S is a finite set of places of a number field F, we denote by $\mathbf{A}_F^S = \prod'_{v \notin S} F_v$, and $F_S := \prod_{v \in S} F_v$; when S consists of the set of places of F above some finite set of places of $\mathbf{Q}$ (for instance the place p) we use the same notation with those places of $\mathbf{Q}$ instead of S (for instance, $F_p = F_{S_p}$). We denote $\mathbf{A}_F^+ := \mathbf{A}_F^\infty \times F^+$.
- we denote by $\psi\colon F\backslash \mathbf{A}_F \to \mathbf{C}^\times$ the standard additive character as in § 1.1.3;
- we denote by G_F the absolute Galois group of a field F;
- if F is a number field, its class number is denoted by
$$h_F := |F^\times \backslash \mathbf{A}_F^{\infty,\times} / \widehat{\mathscr{O}}_F^\times|$$
- for a place v of F, we denote by ϖ_v a fixed uniformiser at v, by $q_{F,v}$ the cardinality of the residue field; we denote $q_{F,p} := (q_{F,v})_{v \in S_p}$;
- the class field theory isomorphism is normalised by sending uniformisers to geometric Frobenii; for F a number field (respectively a local field), we will then identify characters of G_F with characters of $\mathbf{A}_F^\times / F^\times$ (respectively $F^\times$) without further comment;
- If I is finite index set and $x = (x_i)_i$, $y = (y_i)$ are real vectors, we define $(xy)_i = x_i y_i$ and $x^y := \prod_i x_i^{y_i}$ whenever that makes sense.
- we denote by $\mathbf{1}[\cdot]$ the $\{0,1\}$-valued function on logical propositions such that $\mathbf{1}[\phi] = 1$ if and only if ϕ is true;
- if R/R_0 is a ring extension, A is an R_0-algebra, and X is an R_0-scheme, we denote
$$A_R := A \otimes_{R_0} R, \qquad X_R := X \times_{\operatorname{Spec} R_0} R.$$

2.1.2. *Subgroups of* G *and special elements.* — We denote by Z, A, and N respectively the centre, diagonal torus, and upper unipotent subgroup of G; we let $P = AN$ and $P^1 := P \cap \mathrm{Res}_{F/\mathbf{Q}}\mathrm{SL}_2$. We denote by
$$w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in \mathrm{GL}_2(F)$$
or its image in $\mathrm{GL}_2(A)$ for any F-algebra A. For $r \in \mathbf{Z}_{\geq 1}^{S_p}$, we define
$$w_{r_v,v} := \begin{pmatrix} & 1 \\ -\varpi_v^{r_v} & \end{pmatrix} \in \mathrm{GL}_2(F_v), \qquad w_{r,p} := \prod_{v|p} w_{r_v,v} \in \mathrm{GL}_2(F_p),$$
as well as a sequence of compact subgroups
$$U_{v,r_v} := U_1^1(\varpi_v^{r_v}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_{F,v}) \,:\, a-1 \equiv d-1 \equiv c \equiv 0 \pmod{\varpi_v^{r_v}}\} \subset \mathrm{GL}_2(F_v)$$
$$U_{p,r} := \prod_{v \in S_p} U_{v,r_v}.$$

For $\theta \in (\mathbf{R}/2\pi\mathbf{Z})^{S_\infty}$, we denote $r_\theta := \left(\begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix}\right)_v \in \mathrm{SO}(2, F_\infty)$

2.1.3. *Hecke algebras.* — Let $S \subset S'$ be finite sets of non-archimedean places of F and let $U^S = \prod_{v \notin S} U_v \subset \mathrm{GL}_2(\mathbf{A}_F^{S\infty})$ be an open compact subgroup. For each finite set of finite places S, we define the Hecke algebra

$$\mathscr{H}_{U^S}^{S'} := C_c(U^{S'} \backslash \mathrm{GL}_2(\mathbf{A}_F^{S'\infty})/U^{S'}, \mathbf{Q}) = \prod_{v \notin S} \mathscr{H}_{U_v}.$$

It carries an involution

$$T \mapsto T^{\curlyvee} \tag{2.1.1}$$

arising from the map $g \mapsto g^{-1}$ on the group G.

Let $A_p := A(\mathbf{Q}_p) \subset \mathrm{G}(\mathbf{Q}_p)$ be the diagonal torus, and let A_p^+ be the set of $t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ such that $v(t_1) \geq v(t_2)$ for all $v|p$. The involution

$$t \mapsto t^{\curlyvee} := \det(t)^{-1} t \tag{2.1.2}$$

preserves A_p^+. For S a finite set of places of F disjoint from $S_p \cup S_\infty$, we define the ordinary Hecke algebra

$$\mathscr{H}_{U^{Sp}}^{\mathrm{ord}} := \mathscr{H}_{U^{Sp}}^{Sp} \otimes \mathbf{Q}_p[A_p]$$

over $\mathbf{Q}_p$, which will act on spaces of ordinary modular forms. It is endowed with the involution $\curlyvee$ deduced from (2.1.1) and (2.1.2).

2.1.4. *Measures.* — We use the same notation and conventions for Haar measures and integration as in [YZZ12, § 1.6] and [Dis17, § 1.9]. In particular, we have a regularised integration functional

$$\int_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times}^* f(t)\, dt,$$

which satisfies the following.

Lemma 2.1.1. — *Let f be a smooth function on $\mathbf{A}_E^\times$ that is invariant under $E_\infty^\times$. Let $\mu \subset \mathscr{O}_E^\times$ be a finite index subgroup fixing f (under the scaling action). Then*

$$\int_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times}^* \sum_{x \in E^\times} f(xt)\, dt = \frac{2L(1,\eta)}{h_E} [\mathscr{O}_E^\times : \mu] \int_{\mathbf{A}_E^{\infty,\times}} \sum_{\alpha \in \mu} f(\alpha t)\, d^\bullet t,$$

where $d^\bullet t$ is the Haar measure giving volume 1 to $\widehat{\mathscr{O}}_E^\times$.

Proof. — Let $U \subset \mathbf{A}_E^{\infty,\times}$ be any compact open subgroup fixing f. Since both sides are independent of μ, we may assume that $\mu = \mathscr{O}_E^\times \cap U$. By [YZZ12, (1.6.1) and following paragraphs], we have

$$\int_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times}^* f(t)\, dt = \mathrm{vol}(E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times) \fint_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times} f(t)\, dt = \frac{\mathrm{vol}(E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times)}{|E^\times \backslash \mathbf{A}_E^{\infty,\times}/U|} \sum_{t \in E^\times \backslash \mathbf{A}_E^{\infty,\times}/U} f(t). \tag{2.1.3}$$

Now by a coset identity,

$$|E^\times \backslash \mathbf{A}_E^{\infty,\times}/U| = h_E \frac{|\widehat{\mathscr{O}}_E^\times / U|}{[\mathscr{O}_E^\times : \mu]}$$

and by [YZZ12, §1.6.3], $\mathrm{vol}(E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_F^\times) = 2L(1,\eta)$. Hence (2.1.3) equals

$$\frac{2L(1,\eta)}{h_E} \frac{[\mathscr{O}_E^\times : \mu]}{|\widehat{\mathscr{O}}_E^\times / U|} \sum_{t \in E^\times \backslash \mathbf{A}_E^{\infty,\times}/U} f(t)$$

If we compose with the operator $f(\cdot) \mapsto \sum_{x \in E^\times} f(x\cdot) = \sum_{x \in \mu \backslash E^\times} \sum_{\alpha \in \mu} f(\alpha x \cdot)$, we obtain

$$\frac{2L(1,\eta)}{h_E} \frac{[\mathscr{O}_E^\times : \mu]}{|\widehat{\mathscr{O}}_E^\times / U|} \sum_{t \in \mathbf{A}_E^{\infty,\times}/U} \sum_{\alpha \in \mu} f(\alpha t) = \frac{2L(1,\eta)}{h_E} [\mathscr{O}_E^\times : \mu] \int_{\mathbf{A}_E^{\infty,\times}} \sum_{\alpha \in \mu} f(\alpha t)\, d^\bullet t.$$

□

2.2. Modular forms and their q-expansions. — Let $\mathfrak{h} \subset \mathbf{C}$ be the upper half-plane. We view $\mathrm{G}(\mathbf{R})$ as acting on $\mathfrak{h}^{\Sigma_\infty}$ by Möbius transformations, and identify

$$C_\infty^+ := (\mathbf{R}_+ \mathrm{SO}(2,\mathbf{R}))^{\mathrm{Hom}\,(F,\mathbf{R})} \subset \mathrm{G}(\mathbf{R})$$

with the neutral connected component of the stabiliser of $\mathrm{i} := (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathfrak{h}^{\mathrm{Hom}\,(F,\mathbf{R})}$.

2.2.1. *Nearly holomorphic modular forms.* — A complex *nearly holomorphic (Hilbert) modular form* of weight $\underline{w}$, level U, degree $\leq m = (m_\tau)$, is a function

$$f\colon \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$$

satisfying the following two conditions:

1. for all $g \in \mathrm{G}(\mathbf{A})$, $\gamma \in \mathrm{G}(\mathbf{Q})$, $k \in UC_\infty^+$,
$$f(\gamma g k) = j_{\underline{w}}(k_\infty, \mathrm{i})^{-1} f(g),$$
where for $z \in \mathfrak{h}^{\mathrm{Hom}\,(F,\mathbf{R})}$,
$$j_{\underline{w}}((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})_\tau, z_\tau) := (ad-bc)^{(w_0-w)/2}(cz+d)^w.$$
2. There is a Whittaker–Fourier expansion
$$f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = |y| \sum_{a\in F} W_a^\sharp(y)(Y)\,\mathbf{q}^a \tag{2.2.1}$$
for all $y \in \mathbf{A}_F^+$, $x \in \mathbf{A}_F$, where:
   - we have
   $$W_0^\sharp(y) = y_\infty^{(-w_0+w-2)/2} W_0(y), \qquad W_a^\sharp(y) = (ay_\infty)^{(w_0+w-2)/2} W_a(y)$$
   for polynomials
   $$W_a(y) = W_{f,a}(y) \in \mathbf{C}[(T_\tau)_{\tau\colon F\hookrightarrow\mathbf{R}}]$$
   of degree $\leq m_\tau$ in the variables T_τ, evaluated at $Y := (Y_\tau)_{\tau\colon F\hookrightarrow\mathbf{R}}$ with $Y_\tau = (4\pi y_\tau)^{-1}$;
   - we denote
   $$\mathbf{q}^a = \mathbf{q}^a_{y_\infty} := \psi(ax)\psi_\infty(\mathrm{i}ay_\infty).$$

The polynomial $W_a(y)$ only depends on the class of ay modulo $U \cap \left(\begin{smallmatrix} \mathbf{A}_F^{\infty,\times} & \\ & 1 \end{smallmatrix}\right)$, so that we may write

$$W_{f,a}(y) = W_f(ay)$$

for $W_f(a) := W_{f,1}(a)$. We say that f is *cuspidal* if $W_0(y) = 0$ for all y.

If f is nearly holomorphic of degree 0 (that is, $\leq (0,\ldots,0)$), we simply say that f is a (holomorphic) modular form. If $R \subset \mathbf{C}$ is any subring, we denote by

$$S_{\underline{w}}(U,R) \subset M_{\underline{w}}(U,R) \subset N_{\underline{w}}^{\leq m}(U,R)$$

respectively the spaces of cuspidal forms and holomorphic forms of level U and weight $\underline{w}$, and of nearly holomorphic forms of level U, weight $\underline{w}$, and degree $\leq m = (m_\tau)$, such that for all $a \in \mathbf{A}_F^+$, the polynomials $W_f(a)$ have coefficients in R. We write $N_{\underline{w}}(U,R) := \varinjlim_m N_{\underline{w}}^{\leq m}(U,R)$.

2.2.2. *Twisted modular forms.* — A *twisted nearly holomorphic (Hilbert) modular form* of weight $\underline{w}$, level U, degree $\leq m = (m_\tau)$, is a function

$$f\colon \mathrm{GL}_2(\mathbf{A}_F) \times \mathbf{A}_F^\times \to \mathbf{C}$$

satisfying the following two conditions:

1. for all $g \in \mathrm{G}(\mathbf{A})$, $\gamma \in \mathrm{G}(\mathbf{Q})$, $k \in UC_\infty^+$,
$$f(\gamma g k, \det(\gamma)^{-1}u) = j_{\underline{w}}(k_\infty, \mathrm{i})^{-1} f(g,u);$$

2. there is a Whittaker–Fourier expansion

$$f\left(\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right)\right) = |y| \sum_{a\in F} W_a^\sharp(y,u)(Y)\, \mathbf{q}^a \tag{2.2.2}$$

for all $x \in \mathbf{A}_F$ and $y, u \in \mathbf{A}_F$ such that $(uy)_\infty > 0$, where:

$$W_0^\sharp(y,u) = y_\infty^{(w_0+w-2)/2} W_0(y,u), \qquad W_a^\sharp(y,u) = (ay_\infty)^{(w_0+w-2)/2} W_a(y,u)$$

for polynomials

$$W_a(y,u) = W_{a,f}(y,u) \in \mathbf{C}[(T_\tau)_{\tau\colon F\hookrightarrow \mathbf{R}}]$$

of degree $\leq m_\tau$ in the variables T_τ, evaluated at $Y := (Y_\tau)_{\tau\colon F\hookrightarrow\mathbf{R}}$ with $Y_\tau = (4\pi y_\tau)^{-1}$.

If $R \subset \mathbf{C}$ is any subring, we denote by $M_{\underline{w}}^{\mathrm{tw}}(U,R) \subset N_{\underline{w}}^{\mathrm{tw},\leq m}(U,R)$ the spaces of holomorphic and nearly holomorphic forms of level U, weight $\underline{w}$, and degree $\leq m = (m_\tau)$, such that all the polynomials $W_{f,a}(y,u)$ have coefficients in R.

2.2.3. *Contracted product.* — For any open compact subgroup $U_F \subset \widehat{\mathscr{O}}_F^\times$, let

$$\mu_{U_F} := F^\times \cap U_F, \qquad w_{U_F} := |\{\pm 1\} \cap \mu_{U_F}|, \qquad c_{U_F} = \frac{w_U \cdot 2^{[F:\mathbf{Q}]} h_F}{[\mathscr{O}_F^\times : \mu_{U_F}^2]}. \tag{2.2.3}$$

Let $\varphi\colon \mathbf{A}_F^\times \to \mathbf{C}$ be a Schwartz function, invariant under a subgroup of the form $\mu^2_{U'_F} \subset F^\times$ as above. Then the sum

$$\sideset{}{^\star}\sum_{u\in F^\times} \varphi(u) := c_{U_F} \sum_{u \in \mu^2_{U_F^p}\backslash F^\times} \varphi(u) \tag{2.2.4}$$

is well-defined independently of $U_F \subset U'_F$, and for any such choice the support of the sum is finite.

If f_1, f_2 are twisted nearly holomorphic forms, we may thus define a (plain) nearly holomorphic form $f_1 \star f_2$ by

$$f_1 \star f_2(g) := \sideset{}{^\star}\sum_{u\in F^\times} f_1(g,u) f_2(g,u). \tag{2.2.5}$$

2.2.4. *Differential operators.* — We attach to a nearly holomorphic (genuine or twisted) form f the function

$$\begin{aligned} f^{\mathfrak{h}}\colon \mathrm{G}(\mathbf{A}^\infty) \times \mathfrak{h}^{\mathrm{Hom}\,(F,\mathbf{R})} &\to \mathbf{C} \\ (g^\infty, z = g_\infty \mathrm{i}) &\mapsto j_{\underline{w}}(g_\infty, \mathrm{i}) f(g_\infty); \end{aligned}$$

the map $f \mapsto f^{\mathfrak{h}}$ is injective.

The Maass–Shimura differential operators on functions on $\mathfrak{h}^{\mathrm{Hom}\,(F,\mathbf{R})}$ are defined as follows. For $\tau\colon F \hookrightarrow \mathbf{R}$, $k \in \mathbf{Z}$, let

$$\delta_k^{\tau,\mathfrak{h}} := \frac{1}{2\pi i}\left(\frac{k}{2iy_\tau} + \frac{\partial}{\partial z_\tau}\right), \qquad d^\tau := \frac{1}{2\pi i}\frac{\partial}{\partial z_\tau},$$

a differential operator on the upper half-plane $\mathfrak{h}$. For $w, k \in \mathbf{Z}_{\geq 0}^{\mathrm{Hom}\,(F,\mathbf{R})}$, let

$$\delta_w^{k,\mathfrak{h}} := \prod_\tau \delta^{\tau,\mathfrak{h}}_{w_\tau+2k_\tau} \circ \cdots \circ \delta^{\tau,\mathfrak{h}}_{w_\tau+2} \circ \delta^{\tau,\mathfrak{h}}_{w_\tau}, \qquad d^k := \prod_\tau (d^\tau)^{k_\tau}.$$

Then for any ring $\mathbf{Q} \subset R \subset \mathbf{C}$, this operator defines a map

$$\delta_w^k\colon N_{\underline{w}}^{(\mathrm{tw})\leq m}(U,R) \to N^{\leq m+k}_{(\mathrm{tw}),\underline{w}+(0;2k)}(U,R)$$

such that $\delta_w^k(f)^{\mathfrak{h}} = \delta_w^{k,\mathfrak{h}}(f^{\mathfrak{h}})$. (For a proof of the intuitive fact that the archimedean operator $\delta_{\underline{w}}^r$ indeed preserves the rationality properties of finite Whittaker–Fourier coefficients, see [Hid91, Proposition 1.2], whose calculations also apply to the twisted case.)

By [Shi81, (1.16)], for all $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ we have

$$\delta_w^k = \sum_{0 \leq j \leq k} \prod_{\tau \in \Sigma_\infty} \binom{k_\tau}{j_\tau} \frac{\Gamma(w_\tau + k_\tau)}{\Gamma(w_\tau + j_\tau)} (-4\pi y_\tau)^{j_\tau - k_\tau}\, d^j. \tag{2.2.6}$$

If $w \geq 2m+1$, any $f \in N_{\underline{w}}^{(\mathrm{tw}), \leq \mathrm{m}}(U, R)$ can be written uniquely as

$$f = \sum_{0 \leq r \leq m} \delta_{w-2r}^r f_r$$

with $f_r \in M_{\underline{w}+(0;-2r)}^{(\mathrm{tw})}(U, R)$. (The proof in [Shi76, Lemma 7] carries over to our context.) Thus the linear map

$$\begin{aligned} e^{\mathrm{hol}} \colon N_{\underline{w}}^{(\mathrm{tw}), \leq \mathrm{m}}(U, R) &\to M_{\underline{w}}^{(\mathrm{tw})}(U, R) \\ f &\mapsto f_0 \end{aligned} \tag{2.2.7}$$

is well-defined.

2.3. p-adic modular forms. —

2.3.1. *Arithmetic q-expansion.* — The *q-expansion* map

$$f \mapsto (a \mapsto W_{1,f}(a))$$

sends $S_{\underline{w}}(U, \mathbf{C})$ to $\mathbf{C}^{\mathbf{A}_F^+/U_F F_\infty^+}$, where $U_F = U \cap \mathbf{A}_F^{\infty, \times}$. By the q-expansion principle (see [Dis17, Proposition 2.1.1] for a version in our setting), the map is injective. We denote its image by $\mathrm{S}_{\underline{w}}(U, \mathbf{C})$.

If R is any ring admitting embeddings into $\mathbf{C}$, we denote by

$$\mathrm{S}_\bullet(U, R) \subset R^{\mathbf{A}_F^+/U_F F_\infty^+}$$

the set of those sequences $f = (W_a)_a$ such that for any $\iota \colon R \hookrightarrow \mathbf{C}$, the sequence $(\iota W_a)_a$ is the q-expansion of a cuspform

$$f^\iota \in S_\bullet(U, \mathbf{C}) = \bigoplus_{\underline{w}} S_{\underline{w}}(U, \mathbf{C}).$$

By [Hid91, Theorem 2.2 (i)] (together with a consideration of Galois actions mixing the weights), for any such ring R we have $\mathrm{S}_\bullet(U, R) = \mathrm{S}_\bullet(U, \mathbf{Z}) \otimes R$. For more general rings, the previous equality is taken to be the definition of $\mathrm{S}_\bullet(U, R)$.

It is also convenient to attach to $f \in \mathrm{S}_\bullet(U, R)$ and an embedding ι as above the anti-holomorphic cusp form

$$f^{\iota, \mathrm{a}} := \begin{pmatrix} -1_\infty & \\ & 1_\infty \end{pmatrix} f^\iota.$$

2.3.2. *p-adic modular forms.* — Let L be a finite extension of $\mathbf{Q}_p$ splitting F. A *p-adic L-valued (cohomological) weight* $\underline{w} = (w_0, (w_\tau)_{\tau \colon F \hookrightarrow L})$ is a tuple of integers, all having the same parity, such that $w_\tau \geq 1$ for all $\tau \colon F \hookrightarrow L$. If $\iota \colon L \hookrightarrow \mathbf{C}$ is an embedding, the associated complex weight is $\underline{w}^\iota = (w_0, (w_\tau)_{\iota \circ \tau})$. If $\underline{w}$ is a p-adic weight valued in L, we define $\mathrm{S}_{\underline{w}}(U, L)$ to be the set of q-expansions f such that for every $\iota \colon L \hookrightarrow \mathbf{C}$ the expansion $f^{\iota, \psi}$ has weight $\underline{w}^\iota$. The *p-adic q-expansion* of $f = (W_a)_a \in \mathrm{S}_{\underline{w}}(U, L)$ is the sequence

$$f^{(p)} = (W_a^{(p)}), \qquad W_a^{(p)} := a_p^{(w_0 + w - 2)/2} W_a,$$

so that

$$W_a^\sharp(y) := (ay)_\infty^{(w_0 + w^\iota + 2)/2}\, \iota\left((ay_p)^{(-w_0 - w - 2)/2} W_a^{(p)}\right) \tag{2.3.1}$$

is the Whittaker–Fourier coefficient of f^ι as in (2.2.1).

We denote

$$\begin{aligned} \mathrm{S}^{(p)}_{\underline{w}}(U, L) &:= \{f^{(p)} \mid f \in \mathrm{S}^{(p)}_{\underline{w}}(U, L)\} \subset L^{\mathbf{A}_F^+/U_F F_\infty^+}, \\ \mathrm{S}^{(p)}_{\underline{w}}(U^p, L) &:= \varinjlim_n \mathrm{S}^{(p)}_{\underline{w}}(U^p U_{p,n}, L). \end{aligned} \tag{2.3.2}$$

Let $U_F^p = U^p \cap \mathbf{A}_F^{p\infty,\times}$. The space of cuspidal *p-adic modular forms*

$$\mathbf{S}(U^p, L) \subset L^{\mathbf{A}_F^{\infty,\times}/U_F^p}$$

is the completion of $\mathrm{S}^{(p)}_{\underline{w}}(U^p, L)$ for the norm $||(W_a^{(p)})_a|| := \sup_a |W_a^{(p)}|$, for any $\underline{w}$. By a fundamental result of Hida (see [Hid91, paragraph after Theorem 3.1]), the space $\mathbf{S}(U^p, L)$ is independent of the choice of $\underline{w}$. In particular, if L is a Galois over $\mathbf{Q}_p$, this space is stable by the action of $\mathrm{Gal}(L/\mathbf{Q}_p)$ and so it is of the form $\mathbf{S}(U^p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} L$ for a space $\mathbf{S}(U^p, \mathbf{Q}_p)$.

2.3.3. *Nearly holomorphic forms as p-adic modular forms.* — We may attach a p-adic q-expansion to a nearly holomorphic form with coefficients in a p-adic subfield of $\mathbf{C}$.

Let L be a finite extension of $\mathbf{Q}_p$ and let $\underline{w}$ be a p-adic L-valued weight. We say that

$$f = (W_a^{(p)}) \in L^{\mathbf{A}_F^+/F_\infty^+}$$

is a *p-adic nearly holomorphic form* of weight $\underline{w}$ and level $U^p \subset \mathrm{GL}_2(\mathbf{A}_F^{p\infty})$ if the following condition holds. For each $\iota\colon L \hookrightarrow \mathbf{C}$, there exists a nearly holomorphic form

$$f^\iota \in N^{\leq \lfloor (w+1)/2 \rfloor}_{\underline{w}^\iota}(U^p U_{p,n}, \mathbf{C})$$

for some $n \in \mathbf{Z}_{\geq 1}^{S_p}$, whose Whittaker–Fourier polynomials have constant terms satisfying

$$W_{f^\iota, a}(0) = \iota\left(a_p^{(-w_0 - w - 2)/2} W_a^{(p)}\right). \tag{2.3.3}$$

The notion of a p-adic *twisted* nearly holomorphic form is defined similarly by the identity $W_{f^\iota, a}(y, u)(0) = \iota\left((ay)_p^{(-w_0 - w - 2)/2} W_a^{(p)}(y, u)\right)$.

Proposition 2.3.1. — *If f is a p-adic nearly holomorphic cuspform over L of level U^p, then it belongs to the space $\mathbf{S}(U^p, L)$ of p-adic modular cuspforms of level U^p.*

Proof. — This is the first assertion of [Hid91, Proposition 7.3]. ☐

2.3.4. *Hecke operators and ordinary projection.* — The space $N_{\underline{w}}(U, \mathbf{C})$ is endowed with the usual action of $\mathscr{H}_U$. By writing down the effect of this action on Whittaker–Fourier coefficients of cuspforms, this extends to a bounded action of $\mathscr{H}_{U^p}$ on $\mathrm{S}^{(p)}(U^p, \mathbf{Q}_p)$, hence on $\mathbf{S}(U^p)$.

For $t \in A_p^+$ or $y \in \prod_{v|p} \mathscr{O}_{F,v} - \{0\}$, and any $n \in \mathbf{Z}_{\geq 1}^{S_p}$, define the double coset operators

$$\begin{aligned} \mathrm{U}_t &:= [U_{p,n}\, t\, U_{p,n}], \qquad & \mathrm{U}_t^{\circ, \underline{w}} &:= t_1^{2-w} \det(t)^{(-w_0 + w - 2)/2} \mathrm{U}_t, \\ \mathrm{U}_y &:= \mathrm{U}_{\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)}, & \mathrm{U}_y^{\circ, \underline{w}} &:= y^{(-w_0 - w - 2)/2} \mathrm{U}_y \end{aligned} \tag{2.3.4}$$

If L is a finite extension of $\mathbf{Q}_p$, then for all $y \in \prod_{v|p} \mathscr{O}_{F,v} - \{0\}$ we also define the operator on $\mathbf{S}(U^p)$

$$\mathrm{U}_y^\circ \colon \mathbf{S}(U^p, \mathbf{Q}_p) \to \mathbf{S}(U^p, \mathbf{Q}_p)$$
$$W_{\mathrm{U}_y^\circ f}(c) := W_f(cy).$$

This is compatible with the previous definition in the following sense (see [Hid91, (2.2b)], where U_y is denoted by $T(y)$): if f is a p-adic nearly holomorphic form of weight $\underline{w}$ over L, then for all $\iota\colon L \hookrightarrow \mathbf{C}$ we have

$$(\mathrm{U}_y^\circ f)^\iota = \mathrm{U}_y^{\circ, \underline{w}^\iota} f^\iota.$$

By abuse of notation, the superscript $\underline{w}$ will be omitted when understood from context. The ordinary projector is

$$e^{\mathrm{ord}} := \lim_{n\to\infty} (\mathrm{U}_p^\circ)^{n!} \quad \subset \quad \mathrm{End}\,(\mathbf{S}(U^p, L)) \tag{2.3.5}$$

for any tame level U^p and p-adic field L. Its image is denoted by

$$\mathbf{S}^{\mathrm{ord}}(U^p, L) := e^{\mathrm{ord}}\mathbf{S}(\mathbf{U^p}, \mathbf{L}).$$

If π is an automorphic representation of $\mathrm{G}(\mathbf{A})$ over L, then $\pi^{\mathrm{ord}} := \varinjlim_n e^{\mathrm{ord}}\pi^{U_{p,n}}$ is the space of ordinary forms.

If $f_{\mathbf{C}}$ is a complex modular form arising as $f_{\mathbf{C}} = f^\iota$ for a form $f \in \mathrm{S}(L)$ for some finite extension L of $\mathbf{Q}_p$ and some $\iota\colon L \hookrightarrow \mathbf{C}$, we define

$$e^{\mathrm{ord},\iota}(f_{\mathbf{C}}) := (e^{\mathrm{ord}} f)^\iota.$$

2.3.5. *Differential operators after ordinary and holomorphic projections.* — Let L be a finite extension of $\mathbf{Q}_p$, and let f_1, f_2 be p-adic twisted nearly holomorphic forms. For any $\iota\colon L \hookrightarrow \mathbf{C}$, we have

$$[e^{\mathrm{ord}}(f_1 \star d^r f_2)]^\iota = e^{\mathrm{ord},\iota}[e^{\mathrm{hol}}(f_1^\iota \star \delta^r f_2^\iota)]; \tag{2.3.6}$$

the proof of [Hid91, Prop. 7.3] carries over to the twisted case.

2.4. Hida families.

2.4. Hida families. — We gather the fundamental notions concerning Hida families and the associated sheaves of modular forms.

2.4.1. *Weight space.* — Let $U_{F,p}^\circ = \prod_{v|p} U_{F,v}^\circ \subset \mathscr{O}_{F,p}^\times$ be a compact open subgroup (which will be fixed once and for all in § 2.4.4). Let $U_F^p \subset \mathbf{A}_F^{p\infty,\times}$ be a compact open subgroup, and consider the topological groups (with the profinite topology)

$$[Z]_{U_F^p} := Z(F)U_F^p\backslash Z(\mathbf{A}_F^\infty), \qquad [Z]_{U_F^p} \times U_{F,p}^\circ;$$

the latter is isomorphic to $\Delta \times \mathbf{Z}_p^{1+[F:\mathbf{Q}]+\delta_{F,p}}$, where Δ is a finite group and $\delta_{F,p}$ is the p-Leopoldt defect of F. It is embedded into $A(\mathbf{A}_F^\infty)$ by

$$(z, y_p) \mapsto \begin{pmatrix} zy_p & \\ & z \end{pmatrix}.$$

The *weight space* (of tame level U_F^p) is

$$\mathfrak{W} = \mathfrak{W}_{U_F^p} := \mathrm{Spec}\,\mathbf{Z}_p[\![[Z]_{U_F^p} \times U_{F,p}^\circ]\!]_{\mathbf{Q}_p}. \tag{2.4.1}$$

A point $\underline{\kappa} \in \mathfrak{W}$ is identified with the pair of characters

$$\left(\kappa_0 := \kappa_{|[Z]_{U_F^p}}, \qquad \kappa = \underline{\kappa}_{|U_{F,p}^\circ}\right). \tag{2.4.2}$$

We have an involution defined by

$$\underline{\kappa}^\curlyvee(t) := \kappa_0(\det t)^{-1}\underline{\kappa}(t).$$

If $\underline{k}$ is a p-adic weight for G, we say that $\underline{\kappa}$ is *classical of weight* $\underline{k}$ if for all $v|p$,

$$\kappa_0^{\mathrm{sm}}(z_p) := \kappa_0(z_p)z_p^{-k_0}, \qquad \kappa_v^{\mathrm{sm}}(y) := \kappa_v(y)\prod_{\tau|v}\tau(y)^{(-k_0-k_\tau-2)/2}$$

are smooth characters of $F_p^\times$, respectively $U_{F,v}^\circ$, where the product runs over the $\tau \in \Sigma_p$ inducing the place $v \in S_p$. For a classical weight $\underline{\kappa}$, we define

$$\kappa' := \kappa^{\mathrm{sm}}\kappa_{0,p}^{\mathrm{sm},-1} = \kappa^{\curlyvee,\mathrm{sm}}, \tag{2.4.3}$$

a smooth character of $U_{F,p}^\circ$.

We denote by

$$\mathfrak{W}^{\mathrm{cl}} \subset \mathfrak{W}$$

the set of points of classical weight, which has the structure of an ind-étale ind-finite scheme over $\mathbf{Q}_p$. If $\underline{\kappa}$ is classical of weight $\underline{k} = (k_0, k)$, then $\underline{\kappa}^{\curlyvee}$ is classical of weight $\underline{k}^{\vee} = (-k_0, k)$. We let $\mathfrak{W}^{\mathrm{cl}, \geq 2}$ be the set of classical points satisfying $k \geq 2$.

2.4.2. *Hida families.* — Let $\mathbf{S}^{\mathrm{ord,a}}(U^p)$ be the space $\mathbf{S}^{\mathrm{ord}}(U^p)$ endowed with the dualised action $T.f := T^{\curlyvee} f$ of $\mathscr{H}^{\mathrm{ord}}_{U^p}$. Since for every $f \in \mathrm{S}^{(p)}(U^p, \overline{\mathbf{Q}}_p)$ and $\iota \colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ we have

$$T\overline{f^{\iota}} = \overline{(T^{\curlyvee} f)^{\iota}},$$

the space $\mathbf{S}^{\mathrm{ord,a}}(U^p)$ may be identified with the completion of the space of anti-holomorphic ordinary cuspforms. Let

$$\mathbf{T}^{\mathrm{sph,ord}}_{U^p} \subset \mathbf{T}^{\mathrm{ord}}_{U^p} \subset \mathrm{End}\,(\mathbf{S}^{\mathrm{ord,a}}(U_p, \mathbf{Q}_p))$$

be the images of $\mathscr{H}^{\mathrm{sph,ord}}_{U^p}$, $\mathscr{H}^{\mathrm{ord}}_{U^p}$. We let

$$\mathscr{Y}_{\mathrm{G}, U^p} := \mathrm{Spec}\, \mathbf{T}^{\mathrm{sph,ord}}_{U^p}$$

be the *ordinary eigenvariety* for G, which by construction carries a (coherent) sheaf

$$\mathscr{S}^{U^p}$$

whose module of global sections is $\mathbf{S}^{\mathrm{ord,a}}(U^p)$.(8) The space $\mathscr{Y}_{\mathrm{G}, U^p}$, called the *ordinary eigenvariety* for G of tame level U^p, is a union of finitely many irreducible components, called *Hida families* of tame level (dividing) U^p.

Letting $U^p_F := U^p \cap Z(\mathbf{A}^{p\infty})$, we have a *weight-character* map

$$\underline{\kappa}_{\mathrm{G}} \colon \mathscr{Y}_{\mathrm{G}, U^p} \to \mathfrak{W}_{U^p_F}$$

which, when identified with a pair $(\kappa_{\mathrm{G},0}, \kappa_{\mathrm{G}})$ of $\mathscr{O}(\mathscr{Y}_{\mathrm{G}})^{\times}$-valued characters as in (2.4.2), is $\kappa_{\mathrm{G},0}(z) = T(z)$, $\kappa_G(y_p) = \mathrm{U}^{\circ}_{y_p}$. The weight map is finite and flat and it intertwines the involutions $\curlyvee$.

The set of classical points of $\mathscr{Y}_{\mathrm{G}}$ is

$$\mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}} := \mathscr{Y}_{\mathrm{G}} \times_{\mathfrak{W}} \mathfrak{W}^{\mathrm{cl}, \geq 2} \subset \mathscr{X}.$$

If $x_0 \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}$, we denote by π_{x_0} the automorphic representation of $\mathrm{G}(\mathbf{A})$ on which $\mathscr{H}^{\mathrm{sph}}_{U^p}$ acts by the $\mathbf{Q}_p(x)$-character corresponding to x. If $x \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}(\mathbf{C})$ corresponds to $(x_0 \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}, \iota \colon \mathbf{Q}_p(x_0) \hookrightarrow \mathbf{C})$, we denote $\pi_x := \pi^{\iota}_{x_0}$.

2.4.3. *Hida schemes.* — In light of the examples of $\mathfrak{W}$ and $\mathscr{Y}_{\mathrm{G}}$, it will be convenient to introduce a suitable category of spaces.(9) Define the category of *Hida rings* to consist of finite flat $\mathbf{Z}_p[\![X_1, \ldots, X_n]\!]$-algebras A° (for some n) and continuous maps, and the category of *Hida algebras* to be the image of Hida rings under the functor $\otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Define the category of affine Hida schemes to be dual to the category of Hida algebras. A *Hida scheme* is an open subset of an affine Hida scheme. If A°_i are Hida rings (for $i = 1, 2$) and $\mathscr{X}_i = \mathrm{Spec}\,(A^{\circ}_i \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$ (for $i = 1, 2$), we define

$$\mathscr{X}_1 \hat{\times} \mathscr{X}_2 := \mathrm{Spec}\,(A^{\circ}_1 \hat{\otimes} A^{\circ}_2)_{\mathbf{Q}_p},$$

where $\hat{\otimes}$ is the completed tensor product.

2.4.4. *Weight-character map for* H. — Let

$$\mathscr{Y}_{\mathrm{H}} = \mathscr{Y}_{\mathrm{H}, U^p_{\mathrm{H}}} := \mathrm{Spec}\, \mathbf{Z}_p[\![\mathrm{H}(\mathbf{Q}) \backslash \mathrm{H}(\mathbf{A}^{p\infty}) / U^p]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

(8) We have constructed $\mathscr{Y}_{\mathrm{G}}$ based on anti-holomoprhic forms for convenience reasons when considering the interpolation of Petersson products in § 4.1.

(9) The treatment proposed here is minimal and somewhat ad hoc, but it will be sufficient for our purposes. We believe that a more systematic treatment of the geometry of Hida theory should be based on the theory of uniformly rigid spaces developed in [Kap12].

as in (1.1.4). A (Hida) *family* for H is a connected component of $\mathscr{Y}_{\rm H}$.

Fix a sufficiently small open compact subgroup $U_{F,p}^{\circ,\surd} = \prod_{v|p} U_{F,v}^{\circ,\surd} \subset \mathscr{O}_{F,p}^{\times}$ and an injective group homomorphism

$$j''\colon (U_{F,p}^{\circ,\surd}) \to \mathscr{O}_{E,p}^{\times,1} := \{t \in \mathscr{O}_{E,p}^{\times} \mid N_{E_p/F_p}(t) = 1\},$$

and let

$$U_{F,p}^{\circ} := (U_{F,p}^{\circ,\surd})^2 \subset \mathscr{O}_{F,p}^{\times},$$

which is now fixed as promised in § 2.4.1. Let $\sqrt{\ }\colon U_{F,p}^{\circ} \to U_{F,p}^{\circ,\surd}$ be the square root, and let and let $j, j'\colon U_{F,p}^{\circ} \to \mathscr{O}_{E,p}^{\times}$ be the maps[(10)]

$$j'(a) := j''(\sqrt{a})/\sqrt{a}, \qquad j(a) = j'(a)a. \tag{2.4.4}$$

For any open compact $U_E^p \subset \mathbf{A}_E^{p\infty,\times}$ and $U_F^p := U_E^p \cap \mathbf{A}_F^{p\infty,\times}$, define a map

$$\begin{aligned} \underline{\kappa}_{\rm H}\colon\ & \mathscr{Y}_{{\rm H},U_E^p} \to \mathfrak{W}_{U_F^p} \\ & y \mapsto \underline{\kappa}_{\rm H}(y) = \left(\kappa_0 = \chi_{y|[Z]_{U_F^p}}, \kappa := \chi_y \circ j\right). \end{aligned} \tag{2.4.5}$$

The classical points is

$$\mathscr{Y}_{\rm H}^{\rm cl} := \mathscr{Y}_{\rm H} \times_{\mathfrak{W}} \mathfrak{W}^{\rm cl}.$$

Note that if $y \in \mathscr{Y}_{\rm H}$ is a classical point such that $\underline{\kappa}_{\rm H}(y)$ has weight (l_0, l), then χ_y has weight (l_0, l) as defined in the introduction.

2.4.5. *Hida families for* G × H. — These are defined as in § 1.1.4.

2.4.6. *Universal automorphic sheaf on a Hida family.* — Let $\mathscr{X}$ be a Hida family for G (of arbitrary tame level), and let $\mathscr{X}^{\rm cl} := \mathscr{X} \cap \mathscr{Y}_{\rm G}^{\rm cl}$. For each sufficiently small U^p, we may view $\mathscr{X} \subset \mathscr{Y}_{{\rm G},U^p}$ and we define

$$\Pi^{U^p} = \Pi_{\mathscr{X}}^{U^p} := \mathscr{S}_{|\mathscr{X}}^{U^p}.$$

For each $x \in \mathscr{X}^{\rm cl}$, by Hida's Control Theorem (see for instance [Hid91, Corollary 3.3]) and the theory of newforms, we have an isomorphism of $\mathscr{H}_{U^p}$-modules,

$$\Pi_{|x}^{U^p} \cong \pi_x^{U^p,{\rm ord}} := e^{\rm ord}\pi_x^{U^p}. \tag{2.4.6}$$

Let $U_{\mathscr{X}}^p$ be minimal such that $\mathscr{X}$ is a component of $\mathscr{Y}_{{\rm G},U_{\mathscr{X}}^p}$. By [Hid91, §3], there is a unique

$$\mathbf{f}_0 = \mathbf{f}_{0,\mathscr{X}} \in \Pi_{\mathscr{X}}^{U_{\mathscr{X}}^p} \tag{2.4.7}$$

(the *normalised primitive form* over $\mathscr{X}$) whose first coefficient is $(\mathbf{f}_0)_1 = 1$. Any $\mathbf{f} \in \Pi^{U^p}$ can be written as $\mathbf{f} = T\mathbf{f}_0$ for some Hecke operator T supported at the places $v \nmid p\infty$ such that U^p is not maximal.

2.4.7. *Universal Galois sheaf on a Hida family and local-global compatibility.* — Let $\mathscr{X}_{\rm G}$ be a Hida family for G. By results of Hida and Wiles (see [Dis/b, Proposition 3.2.4]), there exist an open subset $\mathscr{X}_{\rm G}' \subset \mathscr{X}_{\rm G}$ containing $\mathscr{X}_{\rm G}^{\rm cl}$ and a locally free sheaf $\mathscr{V} = \mathscr{V}_{\mathscr{X}_{\rm G}'}$ of rank 2, endowed with a Galois action

$$G_F \to {\rm End}_{\mathscr{O}_{\mathscr{X}_{\rm G}'}}(\mathscr{V})$$

such that for all $x \in \mathscr{X}_{\rm G}^{\rm cl}$, the fibre $\mathscr{V}_{|x}$ is the Galois representation attached to π_x by the global Langlands correspondence.

Let S be a finite set of finite places of F, disjoint from S_p, such that for all $v \notin S$ the tame level of $\mathscr{X}_{\rm G}$ is maximal at v. We define

$$\Pi_{\mathscr{X}_{\rm G}}^{Sp} := \varinjlim_{U_S} \Pi_{\mathscr{X}_{\rm G}}^{U^{Sp}U_S},$$

(10) If $v|p$ splits in E, then for $a \in U_{F,v}^{\circ}$ we have $j(a) = (a, 1)$ under some isomorphism $E_v^{\times} \cong F_v^{\times} \times F_v^{\times}$.

which is a finitely generated $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}}[\mathrm{GL}_2(F_S)]$-module. On the other hand, [Dis20, Theorem 4.4.1] attaches to the restriction $\mathscr{V}_v := \mathscr{V}_{|G_{F_v}}$ an $\mathscr{O}_{\mathscr{X}'_{\mathrm{G}}}[\mathrm{GL}_2(F_S)]$-module

$$\Pi(\mathscr{V}_v),$$

which is torsion-free and *co-Whittaker* in the sense of [Dis20, Definition 4.2.2].

Proposition 2.4.1. — *After possibly replacing $\mathscr{X}'_{\mathrm{G}}$ with another open subset $\mathscr{X}''_{\mathrm{G}} \subset \mathscr{X}'_{\mathrm{G}}$ still containing $\mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$, there exists a line bundle $\Pi^{\circ}_{\mathscr{X}'_{\mathrm{G}}}$ over $\mathscr{X}'_{\mathrm{G}}$ with trivial $\mathrm{GL}_2(F_S)$-action, such that*

$$\Pi_{|\mathscr{X}'_{\mathrm{G}}} \cong \Pi^{\circ}_{\mathscr{X}'_{\mathrm{G}}} \otimes \bigotimes_{v \in S} \Pi(\mathscr{V}_v).$$

Proof. — By the local-global compatibility of the Langlands correspondence for Hilbert modular forms (see [Car86] or [Dis/b, Theorem 2.5.1]), for all $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ and all places v, the $\mathrm{GL}_2(F_v)$-representation $\pi_{x,v}$ corresponds, under local Langlands, to the Weil–Deligne representation $V_{x,v}$ attached to $\mathscr{V}_{|x|G_{F_v}}$. Then the result follows from [Dis20, Theorem 4.4.3]. □

2.4.8. *$\mathscr{X}$-adic modular forms.* — Let $\mathscr{X}$ be a Hida scheme endowed with a map $\varphi\colon \mathscr{X} \to \mathscr{Y}_{\mathrm{G}}$. A meromorphic cuspidal *ordinary $\mathscr{X}$-adic modular form* is a sequence

$$f \in \mathscr{O}(\mathscr{X})^{\mathbf{A}^+_F/F^+_\infty} \otimes_{\mathscr{O}(\mathscr{X})} \mathscr{K}(\mathscr{X}),$$

such that for all z in some dense set of closed points $\Sigma \subset \mathscr{X}$, the specialisation $f(z)$ is an ordinary p-adic modular form in $\mathscr{S}^{\curlyvee}_{|\varphi(z)} \otimes_{\mathbf{Q}_p(\varphi(z))} \mathbf{Q}_p(z)$.

Lemma 2.4.2. — *The inclusion j of $\mathscr{S}^{\curlyvee} \otimes_{\mathscr{O}_{\mathscr{Y}_{\mathrm{G}}}} \mathscr{K}(\mathscr{X})$ into the space of meromorphic ordinary $\mathscr{X}$-adic forms is an isomorphism.*

Proof. — Let f be a meromorphic $\mathscr{X}$-adic ordinary form. Up to multiplying by an element of $\mathscr{O}(\mathscr{X})$, we may assume that the coefficients of f belong to a Hida ring A over $B := \mathbf{T}^{\mathrm{sph,ord}}_{\mathbf{Z}_p}$. Let $\mathfrak{p}_z \subset A$ be the ideal corresponding to $z \in \Sigma$. Let $n \in \mathbf{N}$ and let N range among finite subsets of Σ; the filtered system of ideals

$$I_{n,N} := (p^n) + \bigcap_{z \in N} \mathfrak{p}_z$$

forms a fundamental system of neighbourhoods of $0 \in A$, i.e. $A = \varprojlim_{n,N} A/I_{n,N}$. Now by assumption, the image of f in $(A/I_{n,N})^{\mathbf{A}^+_F/F^+_\infty}$ belongs to $\mathbf{S}^{\mathrm{ord}}_{\mathbf{Z}_p} \otimes_B (A/I_{n,N})$, where $\mathbf{S}^{\mathrm{ord}}_{\mathbf{Z}_p}$ is the B-module of integral ordinary p-adic modular forms. It follows that the inclusion j, restricted to integral $\mathscr{X}$-adic forms without poles, is surjective modulo $I_{n,N}$ for all n, N, hence surjective. □

3. Theta–Eisenstein family

In this section, we define the kernel of the Rankin–Selberg convolution giving the p-adic L-function.

3.1. Weil representation.

— We recall the definition of the Weil representation for groups of similitudes; this subsection is largely identical to [Dis17, §3.1].

3.1.1. *Local case.* — Let $V = (V, q)$ be a quadratic space of even dimension over a local field F of characteristic not 2. Fix a nontrivial additive character ψ of F. For simplicity we assume V has even dimension. For $u \in F^\times$, we denote by V_u the quadratic space (V, uq). We let $\mathrm{GL}_2(F) \times \mathrm{GO}(V)$ act on the usual space of Schwartz functions[11] $\mathcal{S}'(V \times F^\times)$ as follows (here $\nu\colon \mathrm{GO}(V) \to \mathbf{G}_m$ denotes the similitude character):

- $r(h)\phi(x,u) = \phi(h^{-1}x, \nu(h)u)$ for $h \in \mathrm{GO}(V)$;
- $r(n(b))\phi(x,u) = \psi(buq(x))\phi(x,u)$ for $n(b) \in N(F) \subset \mathrm{GL}_2(F)$;

[11] The notation is only provisional for the archimedean places, see below.

– $r\left(\begin{pmatrix} a & \\ & d\end{pmatrix}\right)\phi(x,u)=\chi_{V_u}(a)|\tfrac{a}{d}|^{\frac{\dim V}{4}}\phi(at,d^{-1}a^{-1}u)$;

– $r(w)\phi(x,u)=\gamma(V_u)\hat\phi(x,u)$ for $w=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$.

Here $\chi_V=\chi_{(V,q)}$ is the quadratic character attached to V, $\gamma(V,q)$ is a fourth root of unity, and $\hat\phi$ denotes Fourier transform in the first variable with respect to the self-dual measure for the character $\psi_u(x)=\psi(ux)$. We will need to note the following facts (see e.g. [JL70]): χ_V is trivial if V is a quaternion algebra over F or $V=F\oplus F$, and $\chi_V=\eta$ if V is a separable quadratic extension E of F with associated character η.

3.1.2. *Fock model and reduced Fock model.* — Assume that $F=\mathbf{R}$ and V is positive definite. Then we will prefer to consider a modified version of the previous setting. Let the *Fock model* $\mathcal{S}(V\times\mathbf{R}^\times,\mathbf{C})$ be the space of functions spanned by those of the form

$$H(u)P(x)e^{-2\pi|u|q(x)},$$

where H is a compactly supported smooth function on $\mathbf{R}^\times$ and P is a complex polynomial function on V. This space is not stable under the action of $\mathrm{GL}_2(\mathbf{R})$, but it is so under the restriction of the induced $(\mathfrak{gl}_{2,\mathbf{R}},\mathbf{O}_2(\mathbf{R}))$-action on the usual Schwartz space (see [YZZ12, §2.1.2]).

We will also need to consider the *reduced Fock space* $\overline{\mathcal{S}}(V\times\mathbf{R}^\times)$ spanned by functions of the form

$$\phi(x,u)=(P_1(uq(x))+\mathrm{sgn}(u)P_2(uq(x)))e^{-2\pi|u|q(x)}$$

where P_1, P_2 are polynomial functions with rational coefficients.

By [YZZ12, §4.4.1, 3.4.1], there is a surjective quotient map

$$\begin{aligned}\mathcal{S}(V\times\mathbf{R}^\times,\mathbf{C})&\to\overline{\mathcal{S}}(V\times\mathbf{R}^\times)\otimes_{\mathbf{Q}}\mathbf{C}\\ \Phi&\mapsto\phi(x,u)=\overline{\Phi}(x,u)=\int_{\mathbf{R}^\times}\fint_{\mathbf{O}(V)}r(ch)\Phi(x,u)\,dh\,dc.\end{aligned}\tag{3.1.1}$$

We let $\mathcal{S}(V\times\mathbf{R}^\times)\subset\mathcal{S}(V\times\mathbf{R}^\times,\mathbf{C})$ be the preimage of $\overline{\mathcal{S}}(V\times\mathbf{R}^\times)$. For the sake of uniformity, when F is non-archimedean we set $\overline{\mathcal{S}}(V\times F^\times)=\mathcal{S}(V\times F^\times):=\mathcal{S}'(V\times F^\times)$.

3.1.3. *Global case.* — Let $(\mathbf{V},q)$ be an even-dimensional quadratic space over the adèles $\mathbf{A}=\mathbf{A}_F$ of a totally real number field F, and suppose that $\mathbf{V}_\infty$ is positive definite; we say that $\mathbf{V}$ is *coherent* if it has a model over F and *incoherent* otherwise. Given an $\widehat{\mathscr{O}}_F$-lattice $\mathscr{V}\subset\mathbf{V}$, we define the space $\mathcal{S}(\mathbf{V}\times\mathbf{A}^\times)$ as the restricted tensor product of the corresponding local spaces, with respect to the spherical elements

$$\phi_v(x,u)=\mathbf{1}_{\mathscr{V}_v}(x)\mathbf{1}_{\varpi_v^{n_v}}(u),$$

if ψ_v has level n_v. We call such ϕ_v the *standard Schwartz function* at a non-archimedean place v. We define similarly the reduced space $\overline{\mathcal{S}}(\mathbf{V}\times\mathbf{A}^\times)$, which admits a quotient map

$$\mathcal{S}(\mathbf{V}\times\mathbf{A}^\times)\to\overline{\mathcal{S}}(\mathbf{V}\times\mathbf{A}^\times)\tag{3.1.2}$$

defined by the product of the maps (3.1.1) at the infinite places and of the identity at the finite places. The Weil representation of $\mathrm{GO}(\mathbf{V})\times\mathrm{GL}_2(\mathbf{A}^\infty)\times(\mathfrak{gl}_{2,F_\infty},\mathbf{O}(\mathbf{V}_\infty))$ is the restricted tensor product of the local representations.

3.1.4. *The quadratic spaces of interest.* — For a quadratic space $\mathbf{V}=(\mathbf{V},q)$ over $\mathbf{A}_F$, we define $\varepsilon(\mathbf{V})=+1$ (respectively -1) if and only if there exists (respectively does not exist) a quadratic space V over F such that $V\otimes_F\mathbf{A}_F=\mathbf{V}$.

In this paper, we will consider the quadratic spaces $\mathbf{V}=(\mathbf{B},q)$, where $\mathbf{B}$ is a quaternion algebra over $\mathbf{A}_F$, definite at all the archimedean places and split at p, and endowed with an $\mathbf{A}_F$-embedding

$\mathbf{A}_E\hookrightarrow\mathbf{B}$, and $q\colon\mathbf{B}=\mathbf{V}\to\mathbf{A}_F$ is its reduced norm. It has a decomposition

$$\mathbf{V}=\mathbf{V}_1\oplus\mathbf{V}_2$$

where $\mathbf{V}_1=\mathbf{A}_E$ (on which the restriction of q coincides with $N_{E/F}$) and $\mathbf{V}_2$ is the q-orthogonal complement. Thus $\varepsilon(\mathbf{V})=\varepsilon(\mathbf{V}_2)$. We denote by r_1 the restriction of r to a representation of $\mathbf{A}_E^\times=\mathrm{GO}(\mathbf{V}_1)$ on $\overline{\mathcal{S}}(\mathbf{V}_1\times\mathbf{A}_F^\times)$.

For each place v, we define

$$\varepsilon(\mathbf{B}_v)=\varepsilon(\mathbf{V}_v)=\begin{cases}+1 & \text{if } \mathbf{B}_v\cong M_2(F_v)\\ -1 & \text{if } \mathbf{B}_v \text{ is a division algebra.}\end{cases}\tag{3.1.3}$$

We have $\varepsilon(\mathbf{V}):=\prod_v\varepsilon(\mathbf{V}_v)=(-1)^{[F:\mathbf{Q}]}\prod_{v\nmid p}\varepsilon(\mathbf{V}_v)$.

3.2. Theta series. — Let $\phi_1\in\overline{\mathcal{S}}(\mathbf{V}_1\times\mathbf{A}_F^\times)$. We define a twisted automorphic form

$$\theta(g,u,\phi_1):=\sum_{x\in E}r(g)\phi_1(x,u).\tag{3.2.1}$$

It satisfies

$$\theta(zg,u,\phi_1)=\eta(z)\theta(g,u,r(z^{-1},1)\phi_1)\tag{3.2.2}$$

for all $z\in\mathbf{A}^\times$.

For a complex weight $\underline{l}$ for H, let

$$\begin{aligned}\phi_{1,\underline{l},\infty}&:=\otimes_{v|\infty}\phi_{1,\underline{l},v},\\ \phi_{1,\underline{l},v}(t,u)&:=\mathbf{1}_{\mathbf{R}^+}(u)\begin{cases}t^{l_v}|u|^{(-l_0+l_v)/2}e^{-2\pi uq(t)} & \text{if } l_v\geq 0\\ (\overline{t})^{-l_v}|u|^{(-l_0-l_v)/2}e^{-2\pi uq(t)} & \text{if } l_v\leq 0\end{cases},\end{aligned}\tag{3.2.3}$$

and let

$$\theta(g,u,\phi_1^\infty;\underline{l}):=\theta(g,u,\phi_1^\infty\phi_{1,\underline{l},\infty}).$$

Define $|\underline{l}|:=(l_0,(|l_v|)_v)$.

Lemma 3.2.1. — *The series $\theta(g,u,\phi_1^\infty;\underline{l})$ is a twisted holomorphic form of weight $(0,\underline{1})+|\underline{l}|$.*

Proof. — The usual proof that classical theta series are automorphic shows that our θ is twisted automorphic. The archimedean component of the central character is easy to determine by (3.2.2). The weight is computed as in [Xue07, §A1 on p. 350]. □

The Whittaker–Fourier expansion of $\theta(\underline{l})$ is standard: for all $g=\left(\begin{smallmatrix}y&x\\&1\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbf{A}_F)$ with $y\in\mathbf{A}_F^+$,

$$\theta(g,u,\phi_1^\infty;\underline{l})=\sum_{a\in F^\times}\sum_{x\in E^\times\colon uq(x)=a}r(g)\phi_1(x,u)=\eta(y)|y|y_\infty^{\frac{l_0+|l|}{2}}\sum_{a\in F^\times}\sum_{\substack{x\in E^\times\\ uq(x)=a}}\phi_1^\infty(yx,y^{-1}u)\,\mathbf{q}^a.\tag{3.2.4}$$

The following expansion result will be used in § 3.4.

Lemma 3.2.2. — *Let $\chi\colon E^\times\backslash\mathbf{A}_E^\times\to\mathbf{C}^\times$ be a locally algebraic character of weight $\underline{l}$, and let $E(g,u)$ be any twisted modular form such that $E(g,u_\infty u)=E(g,u)$ for all $u_\infty\in F_\infty^+$. Suppose that $\phi_1^\infty(0,u)=0$ for all u. Then for all $g=\begin{pmatrix}y&x\\&1\end{pmatrix}\in\mathrm{GL}_2(\mathbf{A}_F)$, we have*

$$\begin{aligned}&\int^*_{E^\times\backslash\mathbf{A}_E^\times/\mathbf{A}_F^\times}\chi(t)\theta(g,u,r(t)\phi_1^\infty;\underline{l})\star E(g,q(t)u)\,dt\\ &=4|D_{E/F}|^{1/2}\sum_{a\in F^\times}\mathbf{1}_{F_\infty^+}(y_\infty^{-1}a)\eta(y)|y|^{1/2}y_\infty^{\frac{|l|+l_0}{2}}\int_{\mathbf{A}_E^{\infty,\times}}\chi^\infty(t)r_1(t)\phi_1^\infty(y,y^{-1}a)\ E(g,q(t)a)\,d^\bullet t\,\mathbf{q}^a.\end{aligned}$$

Proof. — We may assume that U_F is small enough that $E(u)$ is invariant under $u \in U_F$ and $w_{U_F} = 1$. Taking fundamental domains for $\mu_{U_F}^2 \backslash F^\times$, the expression of interest is

$$c_{U_F}\eta(y)|y|^{1/2} \int^*_{E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}_F^\times} \chi(t) \sum_{u\in\mu_{U_F}^2\backslash F^\times} \sum_{a\in\mu_{U_F}^2\backslash F^\times} \sum_{x\in E^\times} \phi_1(t^{-1}xy, y^{-1}q(t)u)\mathbf{1}[uq(x)=a]E(q(t)u,g)\,dt$$

Since the integrand is invariant under $E_\infty^\times$, by Lemma 2.1.1 with $\mu = \mu_{U_F}$ and a change of variables $a = uq(x)$, this equals

$$\frac{2L(1,\eta)c_{U_F}}{h_E[\mathscr{O}_E^\times : \mu_{U_F}]}\eta(y)|y|^{1/2} \int_{\mathbf{A}_E^{\infty,\times}} \sum_{a\in\mu_{U_F}^2\backslash F^\times} \sum_{\alpha\in\mu_{U_F}}$$

$$\chi^\infty(t\alpha)\phi_1^\infty(t^{-1}\alpha^{-1}y, y^{-1}aq(t)\alpha^2))\mathbf{1}_{F_\infty^+}(y_\infty^{-1}a)y_\infty^{\frac{|l|+l_0}{2}}\; E(g,q(t\alpha)a)\,\mathbf{q}^{\alpha^2 a}\,d^\bullet t.$$

By the invariance properties under U_F, this can be brought into the desired expression by a change of variables $a' = \alpha^2 a$ and the calculation

$$\frac{2L(1,\eta)c_{U_F}}{h_E[\mathscr{O}_E^\times : \mu_{U_F}]} = \frac{L(1,\eta)[\mathscr{O}_E^\times : \mathscr{O}_F^\times]}{h_E/h_F} = 4|D_{E/F}|^{1/2},$$

which follows from the definition of $c_{U_F} =$ (2.2.3) and the class number formula □

3.3. Eisenstein series. — Let $\mathbf{V}_2$ be a 2-dimensional quadratic space over $\mathbf{A}_F$, totally definite at the archimedean places. Let $\phi_2 \in \overline{\mathcal{S}}(\mathbf{V}_2 \times \mathbf{A}_F^\times)$ be a Schwartz function, and let $\xi\colon F^\times\backslash\mathbf{A}_F^\times \to \mathbf{C}^\times$ be a locally algebraic character such that $\xi_\infty(x) = x^{k_0}$ for some integer k_0 and for all $x \in F_\infty^+$. Define the automorphic Eisenstein series[(12)]

$$E_r(g,u,\phi_2,\xi) = \frac{L^{(p\infty)}(1,\eta\xi)}{L^{(p\infty)}(1,\eta)} \sum_{\gamma\in P^1(F)\backslash SL_2(F)} \delta_{\xi,r}(\gamma g w_{r,p}) r(\gamma g)\phi_2(0,u)$$

where (with $s \in \mathbf{C}$)

$$\delta_{\xi,r}(g) := \delta_{\xi,r,0}(g)$$

$$\delta_{\xi,r,s}(g) := \begin{cases} \xi(d)^{-1}|a/d|^{s/2}\psi(k_0\theta) & \text{if } g = \left(\begin{smallmatrix} a & b \\ & d\end{smallmatrix}\right) h \text{ with } h = h^\infty r_\theta \in U_{p,r}K_\infty \\ 0 & \text{if } g \notin PK_0(p^r)K_\infty. \end{cases}$$

(The defining sum is absolutely convergent for $\Re(s)$ sufficiently large, and otherwise it is interpreted by analytic continuation.) It satisfies

$$E_r(zg,u,\phi_2,\xi) = \eta\xi^{-1}(z)E_r(g,u,r(x,1)\phi_2,\xi).$$

3.3.1. *Schwartz function at* ∞. — Let $P_{k_0,k} \in \mathbf{R}[X]$ be the (rescaled) Laguerre polynomial

$$P_{k_0,k}(X) := (2\pi i)^{-k_0}(4\pi)^{-k}(k+k_0)!\sum_{j=0}^{k}\binom{k}{j}\frac{(-X)^j}{j!}. \tag{3.3.1}$$

For $k = (k_0,(k_v)) \in \mathbf{Z}\times\mathbf{Z}_{\geq 0}^{\mathrm{Hom}\,(F,\mathbf{R})}$ such that $k_v + k_0 \geq 0$ for all v, define

$$E_r(g,u,\phi_2^\infty,\xi,k) = E_r(g,u,\phi_2^\infty\phi_{2,\infty,k}^\epsilon,\xi)$$

where $\phi_{2,\infty,k} = \otimes_{v|\infty}\phi_{2,v,k_v}$ with

$$\phi_{2,v,k_v}(x,u) = \mathbf{1}_{\mathbf{R}^+}(u)P_{k_0,k_v}(4\pi uq(x))e^{-2\pi uq(x)}. \tag{3.3.2}$$

The series $E_r(\xi,k)$ belongs to $N^{\leq k}_{(-k_0,k+k_0)}$.

(12) For $k_0 = 0$, this is $L^{(p\infty)}(1,\eta\xi)/L^{(p\infty)}(1,\eta)$ times the series defined in [Dis17].

3.3.2. *Whittaker–Fourier expansion.* — The following standard result is essentially [Dis17, Proposition 3.2.1].

Proposition 3.3.1. — *We have*

$$E_r((\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}), u, \phi_2, \xi) = \sum_{a\in F} W_{a,r}((\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}), u, \phi_2, \xi)\psi(ax)$$

where

$$W_{a,r}(g, u, \phi_2, \xi) = \prod_v W_{a,r,v}(g, u, \phi_{2,v}, \xi_v)$$

with, for each v and $a \in F_v$,

$$W_{a,r,v}(g, u, \phi_{2,v}, \xi_v) \quad = \quad \frac{L^{(p\infty)}(1, \eta_v\xi_v)}{L^{(p\infty)}(1, \eta_v)} \int_{F_v} \delta_{\xi,r,v}(wn(b)gw_{r,v})r(wn(b)g)\phi_{2,v}(0, u)\psi_v(-ab)\, db.$$

Here $L^{(p\infty)}(s, \xi'_v) := L(s, \xi'_v)$ if $v \nmid p\infty$ and $L^{(p\infty)}(s, \xi'_v) := 1$ if $v|p\infty$, and we use the convention that $r_v = 0$ if $v \nmid p$.

We choose convenient normalisations for the local Whittaker functions: let $\gamma_{u,v} = \gamma(\mathbf{V}_{2,v}, uq)$ be the Weil index, and for $a \in F_v^\times$ set

$$W^\circ_{a,r,v}(g, u, \phi_{2,v}, \xi_v) := \gamma_{u,v}^{-1} L^{(p)}(1, \eta_v)W_{a,r,v}(g, u, \phi_{2,v}, \xi_v).$$

Then for the global Whittaker functions we have

$$W_{a,r}(g, u, \phi_2, \xi) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1, \eta)} \prod_v W^\circ_{a,r,v}(g, u, \phi_{2,v}, \xi_v) \tag{3.3.3}$$

if $a \in F^\times$, where $\varepsilon(\mathbf{V}_2) = \prod_v \gamma_{u,v}$ equals -1 if $\mathbf{V}_2$ is coherent or $+1$ if $\mathbf{V}_2$ is incoherent. We similarly define $W^\circ_{0,r}(g, u, \phi_2, \xi)$ by the identity

$$W_{0,r}(g, u, \phi_2, \xi) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1, \eta)} W^\circ_{0,r}(g, u, \phi_2, \xi). \tag{3.3.4}$$

A simple calculation shows that for all v and $a \neq 0$,

$$W^\circ_{a,r,}((\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}), u, \phi_{2,v}, \xi_v) = \eta\xi^{-1}(y)|y|^{1/2} W^\circ_{ay,r,}(1, y^{-1}u, \phi_{2,v}, \xi_v). \tag{3.3.5}$$

We will sometimes drop $\phi_{2,v}$ from the notation.

The following sufficient condition for cuspidality will simplify matters a little later on.

Lemma 3.3.2. — *Assume that there is a place $v \nmid p\infty$, at which ξ_v is unramified, such that*

$$\phi_{2,v}(0, u) = 0 \tag{3.3.6}$$

for all u. Then for all $g = (\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix})$ with $y \in \mathbf{A}^+_F, x \in \mathbf{A}_F$, we have

$$W_{0,r}(g, u, \phi_2, \xi) = 0.$$

Proof. — This is a special case of [YZZ12, Proposition 6.10]. □

3.3.3. *Archimedean Whittaker functions.* — We compute them explicitly based on our explicit choice of Schwartz function.

Lemma 3.3.3. — *Let $v|\infty$, let $\xi_v(x) = x^{k_0}$ for some $k_0 \in \mathbf{Z}$, and let $\phi_{2,v}(x, u) := \mathbf{1}_{\mathbf{R}^+}(u)(uq(x))^k e^{-2\pi uq(x)}$ for some $k \in \mathbf{Z}_{\geq 0}$ with $k \geq -k_0$. Let $a \in \mathbf{R}^\times$. Then*

$$W^\circ_{a,v}(1, u, \phi_2) = \begin{cases} 2(2\pi i)^{k_0} \dfrac{k!}{(k + k_0)!} a^{k+k_0} e^{-2\pi a} & \text{if } a, u > 0 \\ 0 & \text{if } a < 0 \text{ or } u < 0 \end{cases}$$

Proof. — We drop the subscript v. We have

$$wn(b) = \begin{pmatrix} (1+b^2)^{-1/2} & -b(1+b^2)^{-1/2} \\ & (1+b^2)^{1/2} \end{pmatrix} r_{\theta_b}$$

with $e^{i\theta_b} = (i-b)/(1+b^2)^{1/2}$. Then $\delta_{\xi,s}(wn(b)) = i^{k_0}(1+b^2)^{-s/2}(1-ib)^{-k_0}$. Since $L(1,\eta) = \pi^{-1}$ and $\gamma_v = i$, we have

(3.3.7)

$$\begin{aligned} i^{-k_0} W_a^\circ(s,1,u,\Phi) &:= i^{-1}\pi^{-1} \int_{\mathbf{R}} \delta_{\xi,s}(wn(b)) r(wn(b))\Phi(0,u)\psi(-ab)\, db \\ &= \pi^{-1} \int_{\mathbf{R}} (1+b^2)^{-s/2}(1-ib)^{-k_0} \int_{\mathbf{C}} \Phi(x,u)\psi(ubq(x))d_u x\, \psi(-ab) db \\ &= \pi^{-1} \int_{\mathbf{R}} (1+b^2)^{-s/2}(1-ib)^{-k_0} \int_{\mathbf{C}} u^{k+1} q(x)^k e^{-2\pi u q(x)} \psi(ubq(x)) d_1 x\, \psi(-ab) db, \end{aligned}$$

where we recall that $d_u x = |u| d_1 x$ and $d_1 x$ is twice the usual Lebesgue measure. The integral over $\mathbf{C}$ is

$$2u^{k+1} \sum_{j=0}^{k} \int_{\mathbf{R}} \int_{\mathbf{R}} \binom{k}{j} x_1^{2j} x_2^{k-j} e^{-2\pi u(1-ib)(x_1^2+x_2^2)} dx_1 dx_2.$$

Since $\int_{\mathbf{R}} x^{2j} e^{-Ax^2}\, dx = A^{-j-1/2}\Gamma(j+1/2)$, this equals

$$2u^k \sum_{j=0}^{k} \binom{k}{j} \Gamma(j+1/2)\Gamma(k-j+1/2) \cdot (2\pi u)^{-k-1}(1-ib)^{-k-1} = 2^{-k}\pi^{-k} k!\, (1-ib)^{-k-1}$$

by the combinatorial identity (see Appendix A)

$$\sum_{j=0}^{k} \binom{k}{j} \Gamma(j+1/2)\Gamma(k-j+1/2) = \pi k!. \tag{3.3.8}$$

Therefore, when $u > 0$,

$$W_a^\circ(s,1,u,\Phi) = i 2^{-k}\pi^{-k} k! \int_{\mathbf{R}} (1+ib)^{-s/2}(1-ib)^{-(s+2k+2k_0+2)/2} e^{-2\pi i ab} db.$$

The integral is the same one appearing in [YZZ12, bottom of p. 55] with $d = 2+2k+2k_0$. By [YZZ12, Proposition 2.11] (whose normalization differs from ours by $L(1,\eta_v\xi_v) = \pi i$), we find

$$W_a^\circ(0,1,u,\Phi) = i^{k_0} 2^{-k}\pi^{-k-1} k! \frac{(2\pi)^{1+k+k_0}}{\Gamma(1+k+k_0)} a^{k+k_0} e^{-2\pi a} = 2(2\pi i)^{k_0} \frac{k!}{(k+k_0)!} a^{k+k_0} e^{-2\pi a}$$

if $a, u > 0$, as well as simpler formulas implying the desired ones in the other cases. □

We deduce the following. Let

$$Q_{k_0,k}(X) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+k_0)!}{(j+k_0)!} (-X)^{k-j}, \tag{3.3.9}$$

which satisfies $Q_{k_0,k}(0) = 1$.

Proposition 3.3.4. — *Let $v|\infty$, let $a \in \mathbf{R}$, let $k_0 \in \mathbf{Z}$, $k \in \mathbf{Z}_{\geq 0}$ with $k \geq -k_0$. Then for $a \neq 0$ we have*

$$W_{a,v}(0, \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right), u; k) = (-1)^k \eta_v \xi_v^{-1}(y)|y|^{1/2} \cdot 2(ay)^{k+k_0} Q_{k_0,k}((4\pi a y)^{-1}) e^{-2\pi a y}$$

if $ay > 0$, $uy > 0$, and $W_{a,v}(0, \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right), u, \phi_{2,k}) = 0$ otherwise.

Proof. — After recalling the definition of $P_{k_0,k}$ in (3.3.1), by Lemma 3.3.3 and (3.3.5) we find the asserted vanishing and that for $ay, uy > 0$ we have

$$W_{a,v}(0, \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right), u, \phi_{2,k}) = \eta_v \xi_v^{-1}(y)|y|^{1/2} \cdot 2(4\pi)^{-k}(-4\pi)^{-k_0}(k+k_0)! \sum_{j=0}^{k} \binom{k}{j} \frac{(k+k_0)!}{(j+k_0)!} (-4\pi a y)^{j+k_0} e^{-2\pi a y},$$

which is equal to the asserted formula. □

Corollary 3.3.5. — *Let $\xi\colon F^\times\backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ with $\xi(x_\infty) = x_\infty^{k_0}$ for some $k_0 \in \mathbf{Z}$. For each $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ with $k \geq -k_0$, we have*

$$E_r(g, u, \phi_2^\infty, \xi, k) = (-1)_\infty^k \delta^k E_r(g, u, \phi_2^\infty, \xi, 0).$$

Proof. — This follows from Proposition 3.3.4 and (2.2.6). □

3.3.4. *Schwartz function at p.* — Let $U_{F,p}^\circ \subset \mathscr{O}_{F,p}^\times$ be as fixed in § 2.4.4, and let $\kappa_2'\colon U_{F,p}^\circ \to \mathbf{C}^\times$ be a smooth character. We define

$$\phi_{2,\kappa_2'}(x,u) := \mathbf{1}_{\mathscr{V}_{2,p}}(x) \cdot \delta^\circ_{U_{F,p}}(u)\kappa_2'(u), \tag{3.3.10}$$

where $\delta^\circ_{U_{F,p}}(u) := \dfrac{\mathrm{vol}(\mathscr{O}_{F,p}^\times)}{\mathrm{vol}(U_{F,p}^\circ)} \cdot \mathbf{1}^\circ_{U_{F,p}}(u)$. Let

$$E_r(\phi_2^{p\infty}; \xi, \kappa_2', k) := E_r(\phi_2^{p\infty}\phi_{\kappa_2',p}\phi_{2,k,\infty}; \xi), \tag{3.3.11}$$

and denote its normalised Whittaker functions by

$$W^\circ_{a,r,v}(g, u, \phi_2^{p\infty}; \xi, \kappa_2', k);$$

depending on the place v we will drop the unnecessary elements from the notation.

3.3.5. *Non-archimedean Whittaker functions.* —

Proposition 3.3.6. — *Let v be a non-archimedean place of F.*

1. *If $v \nmid p$, then $W^\circ_{a,v,r} = W^\circ_{a,v}$ does not depend on r, and for all $a \in F_v$*

$$W^\circ_{a,v}(1, u, \xi) = |d_v|^{1/2} L(1, \eta_v\xi_v)(1 - \xi_v(\varpi_v)) \sum_{n=0}^\infty \xi_{F,v}(\varpi_v)^n q_{F,v}^n \int_{D_n(a)} \Phi_{2,v}(x_2, u)\, d_u x_2,$$

where $d_u x_2$ is the self-dual measure on $(\mathbf{V}_{2,v}, uq)$ and

$$D_n(a) = \{x_2 \in \mathbf{V}_{2,v} \mid uq(x_2) \in a + p_v^n d_v^{-1}\}.$$

(When the sum is infinite, it is to be understood in the sense of analytic continuation from characters $\xi|\cdot|^s$ with $s > 0$.)

2. *For all finite places v, $|d|_v^{-3/2}|D_v|^{-1/2} W^\circ_{a,v}(1, u, \xi) \in \mathbf{Q}[\xi, \Phi_v]$, and for almost all v we have*

$$|d|_v^{-3/2}|D_v|^{-1/2} W^\circ_{a,v}(1, u, \xi) = \begin{cases} 1 & \text{if } v(a) \geq -v(d_v) \text{ and } v(u) = -v(d_v) \\ 0 & \text{otherwise.} \end{cases}$$

3. *If $v|p$, then*

$$W^\circ_{a,r,v}(1, u; \xi, \kappa_2') = \begin{cases} |d_v|^{3/2}|D_v|^{1/2}\xi_v(-1)\kappa'_{2,v}(u) & \text{if } v(a) \geq -v(d) \text{ and } u \in U^\circ_{F,v}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — See [Dis17, Propostion 3.2.3, Lemma 3.2.4] for parts 1 and 2. For part 3, we drop subscripts v and compute

$$\delta_{r,\xi}(wn(b)w_r) = \xi(-1)\mathbf{1}_{\mathscr{O}_F}(b), \qquad \gamma_u^{-1} r(wn(b))\phi_{2,\kappa_2'}(0,u) = |d_v||D_v|^{1/2}\mathbf{1}_{\mathscr{O}_F^\times}(u)\kappa_2'(u)$$

(the latter if $b \in \mathscr{O}_F$), so that

$$W_a^\circ(1, u; \xi, \kappa_2') = |d_v||D_v|^{1/2}\xi(-1)\mathbf{1}_{\mathscr{O}_F^\times}(u)\kappa_2'(u) \int_{\mathscr{O}_F} \psi(-ab)\, db,$$

which gives the asserted value. □

Corollary 3.3.7. — *For $g = \left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)$ with $y \in \mathbf{A}_F^+$, $x \in \mathbf{A}_F$, we have*

$$E_r(g, u, \phi_2^\infty; \xi, k) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1,\eta)} \eta\xi^{-1}(y)|y|^{1/2} \cdot \left(W_0^\circ(y,u) + \sum_{a\in F^+} 2^{[F:\mathbf{Q}]}(ay_\infty)^{k+k_0} W_{a,r}^{\circ,\infty}(0,1,y^{-1}u,\phi_2,\xi) Q_{k_0,k}((4\pi a y_\infty)^{-1})\, \mathbf{q}^a \right)$$

where $W_{a,r}^{\circ,\infty}(0,1,u,\phi_2;\xi) = \prod_{v\nmid\infty} W_{a,r,v}^\circ(0,1,y^{-1}u,\phi_{2,v};\xi)$.

Let $\phi_2^{p\infty}$ satisfy (3.3.6), so that by Lemma 3.3.2, the corresponding Eisenstein series is cuspidal. For ξ a locally algebraic p-adic character of $\mathbf{A}_F^\times$ of weight k_0, consider the (bounded) sequence of coefficients in $\mathbf{Q}_p(\xi)$

$$\mathrm{E}_r(u, \phi_2^\infty, \xi, k) = \left(\lambda \cdot \eta\xi^{-1}(y) 2^{[F:\mathbf{Q}]}(by_p)^{k+k_0}|D_F|^{1/2}|D_E|^{1/2} W_{by}^{\circ,\infty}(1, y^{-1}u, \xi)\right)_{b\in F^+}.$$

where $\lambda = -\varepsilon(\mathbf{V}_2)/|D_{E/F}|^{1/2}L^{(p)}(1,\eta)$. This is the p-adic expansion attached to E_r. Analogously to Corollary 3.3.5, we have

$$\mathrm{E}(u, \phi_2, \xi, k) = (-1)_\infty^k d^k \mathrm{E}(u, \phi_2, \xi, 0). \tag{3.3.12}$$

3.3.6. *p-adic interpolation of Whittaker functions.* —

Proposition 3.3.8. — *Let $v \nmid p\infty$. For each $a \in F_v^\times$, $y \in F_v^\times$, and $\phi_{2,v} \in \mathcal{S}(\mathbf{V}_2^{p\infty} \times \mathbf{A}^{p\infty,\times}, \mathbf{Q}_p)$ there is a meromorphic function*

$$\mathscr{W}_{a,v}^\circ(y, u, \phi_{2,v}) \in \mathscr{K}(\mathscr{Y}_v),$$

regular if $\phi_{2,v}$ is standard and otherwise with possible poles at those ξ_v such that $L(1,\eta_v\xi_v) = 0$, satisfying

$$\mathscr{W}_{a,v}^\circ(y,u,\phi_{2,v};\xi_v) = |d_v|^{-3/2}|D_v|^{-1/2} W_{a,r,v}^\circ(\left(\begin{smallmatrix} y_v & \\ & 1\end{smallmatrix}\right), u, \phi_{2,v}(\xi_v), \xi_v)$$

for all $\xi_v \in \mathscr{Y}_v(\mathbf{C})$ whose underlying scheme point is not a pole.

Proof. — Part 1 is proved as in [Dis17, Lemma 3.3.1], except that we write the arbitrary $\phi_{2,v} = c\phi_{2,v}^\circ + \phi_{2,v}'$ without the extra factor of equation (3.3.2) *ibid.* Then the argument shows that (only) when $\phi_{2,v}' \neq 0$, there may be a pole controlled by $L(1, \eta_v\xi_v)$. □

3.4. Theta-Eisenstein family. — Fix a compact open subgroup $U^p \subset \mathbf{B}^{\infty\times}$ (which will be usually omitted from all the notation), and let $U_F^p := U^p \cap \mathbf{A}^{p\infty,\times}$. Let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty,\times})$ be a Schwartz function fixed by U^p. Let $\xi\colon F^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ be a locally algebraic character fixed by U_F^p and such that $\xi(x_\infty) = x_\infty^{k_0}$ for some $k_0 \in \mathbf{Z}$, and let $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ satisfy $k_v + k_0 \geq 0$ for all v.

We fix a choice of a Schwartz function in $\mathcal{S}(\mathbf{V}_{2,p} \times F_p^\times)$ as follows. Let $U_{F,p}^\circ \subset \mathscr{O}_{F,p}^\times$ be as fixed in § 2.4.4. For $r \in \mathbf{Z}_{\geq 1}^{S_p}$ and $\kappa_1'\colon F_p^\times \to \mathbf{C}^\times$ a smooth character, we define

$$\phi_{1,r,\kappa_1',p}(x,u) = \delta_{1,r,p}(x)\mathbf{1}_{U_{F,p}^\circ}\kappa_1'(u) := \prod_{v|p} \frac{\mathrm{vol}(\mathscr{O}_{E,v}, dt)}{\mathrm{vol}(1+\varpi_v^{r_v}\mathscr{O}_{E,v}, d^\times t)} \mathbf{1}_{1+\varpi_v^{r_v}\mathscr{O}_{E,v}}(x_{1,v})\mathbf{1}_{U_{F,p}^\circ}(u)\kappa_1'(u). \tag{3.4.1}$$

For $t \in \mathbf{A}_E^\times$, $r \geq \underline{1}$, and $\phi^{p\infty} = \phi_1^{p\infty} \otimes \phi_2^{p\infty}$, define a form in $N^k_{(l_0-k_0, 2+l+k_0+2k)}(\mathbf{C})$ by

$$I_r(t, \phi^{p\infty}; \kappa_1', \underline{l}, \xi, \kappa_2', k) := |D_F|^{-1/2} \cdot \theta(u, r_1(t,1)\phi_1^{p\infty}\phi_{1,\kappa_1',r,p}; \underline{l}) \star E_r(q(t)u, \phi_2^{p\infty}; \xi, \kappa_2', k).$$

where the product $\star$ is (2.2.5).

Fix a compact open subgroup $U_{\mathrm{H}}^p \subset \mathbf{A}_E^{p\infty,\times}$ (which will be omitted from all the notation). Let $\chi\colon E^\times\backslash\mathbf{A}_E^\times \to \mathbf{C}^\times$ be a locally algebraic character of weight $\underline{l}$ fixed by U_{H}^p. and assume that for all $w|v|p$, the integer $r_v \geq 1$ is greater than the conductors of χ_w, ξ_v, $\kappa_{2,v}'$. Then we define

$$I(\phi^{p\infty}; \chi, \xi, \kappa_2', k) := \int^*_{E^\times\backslash\mathbf{A}_E^\times/\mathbf{A}_F^\times} \chi(t) I_r(t, \phi^{p\infty}; \kappa_{1,\chi,p}', \underline{l}, \xi, \kappa_2', k)\, dt, \tag{3.4.2}$$

which does not depend on the choice of r; here $\kappa'_{1,\chi,p}$ is as in (2.4.3), namely

$$\kappa'_{1,\chi,p} := \chi_p \circ j' \tag{3.4.3}$$

for $j'\colon U^\circ_{F,p} \to \mathscr{O}^\times_{E,p}$ as in (2.4.4)

Lemma 3.4.1. — *For each $c \in \mathbf{A}_F^+$ satisfying $v(c) \geq 1$ for some $v|p$, the $c^{\rm th}$ Whittaker–Fourier coefficient of $I(\phi^{p\infty}; \chi, \xi, \kappa'_2, k)$ is*

$$W_I^\sharp(c) = \frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{L^{(p)}(1,\eta)} \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}} \cdot \sum_{a/c\in F,\ 0<a/c<1} \chi^\infty(a) a_\infty^{(|l|+l_0)/2} \xi^{\infty,-1}(c-a)(a-c)^k_\infty$$
$$\prod_{v\nmid p\infty} J_v(a,c,\phi_v;\chi_v,\xi_v) \prod_{v|p} J_v(a,c;\chi_v,\xi_v,\kappa'_{2,v}).$$

Here, taking $\phi_{1,v} = \phi_{1,\kappa'_{1,\chi,v},r}$ with $\kappa'_{1,\chi,v} := \chi_v \circ j_v$ and $\phi_{2,v} = \phi_{2,\kappa'_{2,v}}$ if $v|p$, we let

$$J_v(a,c,(\phi_v);\chi_v,\xi_v,(\kappa'_{2,v})) := \int_{E_v^\times} \chi_v(t) r_1(t)\phi_{1,v}(1,a) W^\circ_{c-q(t)a,v}(1,q(t)a,\phi_{2,v};\xi,\kappa'_{2,v})\, d^\bullet t_v, \tag{3.4.4}$$

for the measure $d^\bullet t_v$ giving volume 1 to $\mathscr{O}^\times_{E,v}$.

Proof. — We lighten some of the notation. The assumptions of Lemma 3.2.2 are satisfied, therefore for $c \in F^+$ and $g = \left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)$ with $y \in \mathbf{A}_F^+$, $x \in \mathbf{A}_F$,

$$\begin{aligned} I(g;\chi,\xi,k) &= \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}} \eta(y)|y|^{1/2} y_\infty^{\frac{|l|+l_0}{2}} \sum_{a\in F^+} \int_{\mathbf{A}_E^{\infty,\times}} \chi^\infty(t) r_1(t)\phi_1^\infty(y,y^{-1}a)\ E(g,q(t)a)\, d^\bullet t\, \mathbf{q}^a \\ &= \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}} |y|^{1/2}\eta\chi(y) y_\infty^{(|l|-l_0)/2} \sum_{a\in F^+} \int_{\mathbf{A}_E^{\infty,\times}} \chi^\infty(t) r_1(t)\phi_1^\infty(1,ay)\ E(g,q(t)ay^2)\, d^\bullet t\, \mathbf{q}^a \end{aligned}$$

Now for $\lambda = -\varepsilon(\mathbf{V}_2)/L^{(p)}(1,\eta)$, by Corollary 3.3.7 we have

$$E(g,q(t)ay^2) = \lambda{\cdot}|y|^{1/2}\eta\xi^{-1}(y) \left(W_0(y,u) + (-1)^k_\infty \sum_{b\in F^+} 2^{[F:\mathbf{Q}]} (by_\infty)^{k+k_0} Q_{k_0,\underline{k}}((4\pi b y_\infty)^{-1}) W^{\circ,\infty}_{by}(1,q(t)ay,\xi)\, \mathbf{q}^b \right)$$

so that

$$I(g;\chi,\xi,k) = |y| \sum_{c\in F^\times} W^\sharp_{I,c}(y)\mathbf{q}^c$$

for some coefficients $W^\sharp_{I,c}(y)$, which we now explicitly calculate if c satisfies $v(c) \geq 1$ for some $v|p$. Under this condition, we have

$$\begin{aligned} W^\sharp_{I,c}(y) = \frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{L^{(p)}(1,\eta)} \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}} \sum_{\substack{a\in F\\ 0<a<c}} (ay)_\infty^{(|l|-l_0)/2} \int_{\mathbf{A}_E^{\infty,\times}} \chi(ay)\chi^\infty(t) r_1(t)\phi_1^\infty(1,ay) \\ \cdot (-1)^k_\infty \xi^{-1}((c-a)y)((c-a)y_\infty)^{k+k_0} W^{\circ,\infty}_{(c-q(t)a)y}(1,q(t)ay,\xi) Q_{k_0,\underline{k}}((4\pi(c-a)y_\infty)^{-1})\, dt. \end{aligned}$$

where we have noted that, since we have assumed $v(c) \geq 1$ and the choice of $\phi_{1,p}$ implies $v(a) = 0$, the constant term (corresponding to $a = c$) of the Eisenstein series does not contribute. Finally, we rewrite the resulting formula with c in place of cy. □

Lemma 3.4.2. — *Let $a, c \in \mathbf{A}_F^{\infty,\times}$, $\phi_2 \in \mathcal{S}(\mathbf{V}_2^{p\infty} \times \mathbf{A}_F^{p\infty,\times})$, and for locally algebraic characters $\xi\colon F^\times\backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$, $\chi\colon E^\times\backslash\mathbf{A}_E^\times \to \mathbf{C}^\times$, consider the terms $J_v(a,c;\chi_v,\xi_v) = $ (3.4.4).*

1. *For $v|p$, if $v(c) \geq 1$ then*

$$|d_v|^{-3/2}|D_v|^{-1/2} \cdot J_v(a,c;\chi_v,\xi_v,\kappa'_{2,v}) = \xi_v(-1)\mathbf{1}_{U^\circ_{F,v}}(a)\kappa'_{1,\chi,v}\kappa'_{2,v}(a).$$

2. For all but finitely many $v \nmid p$,

$$|d_v|^{-3/2}|D_v|^{-1/2}J_v(a,c,\phi_{2,v};\chi_v,\xi_v)=1.$$

3. For all $v\nmid p\infty$, there is a function $\mathbf{J}_v(a,c)\in\mathscr{O}(\mathscr{Y}_v\times\mathscr{Y}_{\mathrm{H},v})$ such that for all $\chi_v,\xi_v\in\mathscr{Y}_v\times\mathscr{Y}_{\mathrm{H},v}(\mathbf{C})$,

$$\mathbf{J}_v(a,c,\phi_v)(\chi_v,\xi_v)=|d_v|^{-3/2}|D_v|^{-1/2}J_v(a,c,\phi_{2,v};\chi_v,\xi_v)=1;$$

Proof. — Part 1 follows from the definitions and Proposition 3.3.6.3. Part 2 follows from Proposition 3.3.6.2 and a simple calculation. Finally, since the integrand in (3.4.4) is compactly supported, part 3 follows from Proposition 3.3.8. □

We lighten the notation by occasionally dropping $\phi^{p\infty}$. Let

$$\lambda:=\frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{|D_{E/F}|^{1/2}L^{(p)}(1,\eta)}\in\mathbf{Q}^\times. \tag{3.4.5}$$

For the sake of simplicity, we momentarily introduce the assumption that the weight $\underline{l}$ of χ satisfies $l\geq 0$. We will see in Corollary 3.4.5 that this does not affect in our main construction.

Proposition 3.4.3. — *Let $\chi\in\mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}$ have weight $\underline{\kappa}_1=\underline{\kappa}_{\mathrm{H}}(\chi)$, and let $\underline{\kappa}_2=(\kappa_{2,0},\kappa_2)\in\mathfrak{W}^{\mathrm{cl}}$. Let $U_F^\circ:=U_F^pU_{F,p}^\circ$. Write $\xi=\kappa_{2,0}^{-1}$ and*

$$\begin{aligned}\kappa_{1,0,p}&=\kappa_{1,0,p}^{\mathrm{sm}}(\cdot)^{l_0}, & \kappa_{2,0,p}&=\kappa_{2,0,p}^{\mathrm{sm}}(\cdot)^{-k_0}\\ \kappa_1&=\kappa_i^{\mathrm{sm}}(\cdot)^{(l+l_0)/2}, & \kappa_2&=\kappa_2^{\mathrm{sm}}(\cdot)^{k}.\end{aligned} \tag{3.4.6}$$

Assume that $l_\tau,k_\tau,k_\tau+k_0\geq 0$ for all $\tau\in\Sigma_p$. Let $\kappa_i'\colon U_{F,p}^\circ\to\mathbf{C}^\times$ be as in (2.4.3). Let $n\in\mathbf{N}$ be sufficiently large (depending on ξ,χ). The coefficients

$$W((\mathrm{U}^\circ)_p^{n!}\,\mathrm{I},c)(\chi,\underline{\kappa}_2):=\lambda\cdot\sum_{\substack{a/c\in F\\ 0<a/c<p^{n!}}}\mathbf{1}_{U_{F,p}^\circ}(a_p)\kappa_1(a_p)\kappa_2((a-p^{n!}c)_p)\cdot\kappa_{1,0}^{p\infty}(a)\kappa_{2,0}^{p\infty}(-a)\prod_{v\nmid p}\mathbf{J}_v(a,c;\xi_v,\chi_v), \tag{3.4.7}$$

for $c\in\mathbf{A}_F^+$ with $v(c)\geq 0$ for all $v|p$, define a p-adic modular form

$$(\mathrm{U}_p^\circ)^{n!}\,\mathrm{I}(\chi,\underline{\kappa}_2)\in\mathbf{S}(\mathbf{Q}_p(\chi,\underline{\kappa}_2))$$

such that for every $\iota\colon\mathbf{Q}_p(\chi,\underline{\kappa}_2)\hookrightarrow\mathbf{C}$, we have

$$(\mathrm{U}_p^\circ)^{n!}\mathrm{I}(\chi,\underline{\kappa}_2)^\iota=|D_E|^{1/2}|D_F|(\mathrm{U}_p^\circ)^{n!}I(\chi^\iota,\xi^\iota,\iota\kappa_2',k).$$

Moreover, if $\phi_2^{p\infty}$ satisfies (3.3.6), then

$$e^{\mathrm{ord}}\mathrm{I}(\chi,\underline{\kappa}_2)=e^{\mathrm{ord},\iota}[e^{\mathrm{hol}}(|D_E|^{1/2}|D_F|(\mathrm{U}_p^\circ)^{n!}I(\chi^\iota,\xi^\iota,\iota\kappa_2',k)]. \tag{3.4.8}$$

Proof. — From Lemmas 3.4.1, 3.4.2, we find an expression which can be brought into the above form, after replacing $p^{n!}c$ with c whenever it occurs as the argument of a smooth function.

The second assertion follows from (2.3.6), Corollary 3.3.5 and (3.3.12). □

Let $\mathbf{J}^p(a,c;\chi,\xi):=\prod_{v\nmid p}\xi_v^{-1}(a)\chi_v(-a)\mathbf{J}_v(a,c;\chi_v,\xi_v)$. It is easy to verify, using only that $\mathbf{J}^p$ is a Schwartz function of $a\in\mathbf{A}_F^{p\infty,\times}$, that the Riemann sums

$$\mu_n(\mathbf{J}^p,c)(\varphi):=\sum_{a/c\in F^\times}\varphi(a)\cdot\mathbf{1}[0<a/c<p^{n!}]\mathbf{1}[a\in U_{F,p}^\circ]\cdot\mathbf{J}^p(a,c;\chi,\kappa_{2,0}^{-1}),$$

for $\varphi\colon U_{F,p}^\circ\to\mathbf{Q}_p(\kappa_2,\xi)$ locally constant, converge to a measure

$$\mu(\mathbf{J}^p,c;\chi^{p\infty},\underline{\kappa}_2^{p\infty})$$

(bounded distribution) on $U_{F,p}^\circ$ valued in $\mathbf{Q}_p(\chi,\underline{\kappa}_2)$. Then by the same argument of the standard result in [Kob77, Theorem 6 on p. 39], any continuous $\mathbf{Q}_p(\chi,\underline{\kappa}_2)$-valued is integrable for this measure. The same holds with $\mathbf{Q}_p(\chi, \underline{\kappa}_2)$ replaced by $\mathscr{O}(\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W})$ and $\chi, \underline{\kappa}_2$ by the universal characters, $\mathbf{J}^p$ by the universal function; for this universal situation, we will use the same notation without $\chi, \underline{\kappa}_2$.

Corollary 3.4.4. — *Let S^{bad} be the set of places of F at which $\phi^{p\infty}$ is not the standard Schwartz function. Denote $\kappa_1 = \kappa_{\mathrm{H}}(\chi_{\mathrm{univ}})$. Assume that $\phi_2^{p\infty}$ satisfies* (3.3.6). *The sequence of coefficients*

$$\mathbf{I}^{\mathrm{ord}}(\chi, \underline{\kappa}_2) := (W(\mathbf{I}^{\mathrm{ord}}, c)(\chi, \underline{\kappa}_2))_{c \in \mathbf{A}_F^{\infty,+}, c_p \in \mathscr{O}_{F,p}},$$
$$W(\mathbf{I}^{\mathrm{ord}}, c)(\chi, \underline{\kappa}_2) := \lambda \cdot \int_{U_{F,p}^{\circ}} \kappa_1 \kappa_2(a)\, d\mu(\mathbf{J}^p, c)(a) \in \mathscr{K}(\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W}),$$

has poles controlled by $\prod_{v \in S^{\mathrm{bad}}} L(1, \eta_v \kappa_{2,0,v}^{-1}) = 0$, and it satisfies the following property. For all $\chi \in \mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}(\mathbf{C})$, $\underline{\kappa}_2 \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ with underlying numerical weights $\underline{l}$, $\underline{k}$ such that $l_\tau, k_\tau, k_\tau + k_0 \geq 0$ for all τ, we have

$$\mathbf{I}^{\mathrm{ord}}(\chi, \underline{\kappa}_2) = e^{\mathrm{ord},\iota} |D_E|^{1/2} |D_F| e^{\mathrm{hol}} I(\chi^\iota, \xi^\iota, \iota\kappa_2', k), \tag{3.4.9}$$

where $\iota\colon \mathbf{Q}_p(\chi, \underline{\kappa}_2) \hookrightarrow \mathbf{C}$ is the embedding attached to the complex geometric point $(\chi, \underline{\kappa}_2)$.

Proof. — This follows from Proposition 3.4.3 and the previous discussion. The simplification in the argument of κ_2 in the interpolated coefficient (3.4.7) is justified by the fact that $\kappa_2(a - p^{n!}c) - \kappa_2(a) \to 0$ uniformly in a, and that the expression of interest is a bounded function of $\kappa_2(\cdot)$. □

Consider the weight map

$$(\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}}) \to \mathfrak{W}$$
$$(x, y) \mapsto \kappa_x^{\Upsilon}.$$

Recycling notation (in a way that should cause no confusion), define an ordinary meromorphic $(\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}})$-adic modular cuspform by

$$\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}, x, y) := \mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}, \chi_y, \underline{\kappa}_x^{\Upsilon} \underline{\kappa}_y^{-1}), \tag{3.4.10}$$

where the right-hand side is the form of Corollary 3.4.4.

Corollary 3.4.5. — *Assume that $\phi_2^{p\infty}$ satisfies* (3.3.6). *For all $x \in \mathscr{Y}_{\mathrm{G}}^{\mathrm{cl}}(\mathbf{C})$ and $y \in \mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}(\mathbf{C})$, of weights $\underline{w}$, $\underline{l}$ such that for all $\tau \in \Sigma_p$,*

$$|l_\tau| \leq w_\tau - 2, \qquad |w_0 + l_0| \leq w_\tau - 2 - |l_\tau|,$$

we have

$$\mathbf{I}^{\mathrm{ord}}(x, y) = e^{\mathrm{ord},\iota} |D_E|^{1/2} |D_F| e^{\mathrm{hol}} I(\chi^\iota, \xi_{x,y}^\iota, \iota\kappa_2', k_{x,y}), \tag{3.4.11}$$

where

- *$\xi = \omega_x \omega_y$, whose weight we denote by k_0;*
- *$k = (w - 2 - |l| - k_0)/2$;*
- *$\kappa_2' = \kappa'_{\pi_x^\vee} \cdot \kappa_y'^{-1}$;*
- *$\iota\colon \mathbf{Q}_p(x, y) \hookrightarrow \mathbf{C}$ is the embedding attached to the complex geometric point (x, y).*

Proof. — The interpolation property at characters satisfying $l \geq 0$ follows from Proposition 3.4.3 via Corollary 3.4.4; the same argument also goes through without the assumption $l \geq 0$, since the weight of the chosen theta-Eisenstein series does not depend on $\underline{l}$ (or y) but only on x. The inequalities on the weights come from the conditions $k, k + k_0 \geq 0$. □

4. Zeta integrals

In this final section, we interpolate global and local (away from $p\infty$) zeta integrals, compute the archimedean and p-adic integrals, and construct the p-adic L-function.

As preliminary, we recall gamma factors introduced in the introduction. Let F_v and L be p-adic fields. The (inverse) Deligne–Langlands gamma factor of a potentially semistable representation ρ of $\mathrm{Gal}(\overline{F}_v/F_v)$ over L, with respect to a nontrivial character $\psi_v\colon F_v \to \mathbf{C}^\times$ and an embedding $\iota\colon L \hookrightarrow \mathbf{C}$, is defined as

$$\gamma(\iota\rho, \psi_v)^{-1} := \frac{L(\iota\mathrm{WD}(\rho))}{\varepsilon(\iota\mathrm{WD}(\rho), \psi_v)L(\iota\mathrm{WD}(\rho^*(1)))},$$

where $\iota\mathrm{WD}$ is Fontaine's functor [Fon94] to complex Weil–Deligne representations. It satisfies

$$\gamma(W, c\psi_v)^{-1} = \det W(c)^{-1}\gamma(W, \psi_v)^{-1}. \tag{4.0.1}$$

We also define $\gamma(s, W, \psi_v) := \gamma(W \otimes |\cdot|^s, \psi_v)$.

Let $\mathbf{Q}_p^{\mathrm{ab}}$ be the abelian closure of $\mathbf{Q}_p$. If W is 1-dimensional corresponding to a smooth character $\chi\colon F_v \to \mathbf{Q}_p^{\mathrm{ab},\times}$, then for any $\sigma \in G_{\mathbf{Q}_p}^{\mathrm{ab}}$ corresponding to $a \in \mathbf{Q}_p^\times$ under the reciprocity map, we have

$$\gamma(\chi, \psi_v)^\sigma = \chi(a)\gamma(\chi^\sigma, \psi_v). \tag{4.0.2}$$

For now until the final § 4.5.2, we fix an embedding

$$\iota^{\mathrm{ab}}\colon \mathbf{Q}_p^{\mathrm{ab}} \hookrightarrow \mathbf{C}, \tag{4.0.3}$$

by which we identify the standard character $\psi^\infty\colon \mathbf{A}_F^\infty \to \mathbf{C}$ with one valued in $\mathbf{Q}_p^{\mathrm{ab}}$ (still denoted by ψ^∞).

4.1. Petersson product. — Let π be an ordinary automorphic representation of $\mathrm{G}(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p^{\mathrm{ab}}$. For $v|p$, let $\omega_{\pi,v}$, $\alpha_{\pi,v}\colon F_v^\times \to L^\times$ be the central character and, respectively, U_v-eigencharacter of π_v.

Define $\gamma(\mathrm{ad}(V_{\pi,p})(1)^{++}, \psi_p) := \prod_{v|p}\gamma(\mathrm{ad}(V_{\pi,v})(1)^{++}, \psi_v)$, where $\mathrm{ad}(V_{\pi,p})(1)^{++}$ is the character $\omega_{\pi,v}^{-1}\alpha_{\pi,v}^2|\cdot|^2$ of $F_v^\times$. For $\iota\colon L \hookrightarrow \mathbf{C}$, let

$$e_p(\mathrm{ad}(V_{\pi^\iota})(1)) := \frac{\gamma(\iota\mathrm{ad}(V_{\pi,p})(1)^{++}, \psi_p)^{-1}\zeta_{F,p}(2)}{L(1, \pi_p^\iota, \mathrm{ad})}.$$

Let $(,)$ denote the Petersson pairing

$$(f, f') := \int_{\mathbf{A}_F^\times\backslash\mathrm{GL}_2(\mathbf{A}_F)} f(g)f'(g)\, dg$$

of automorphic forms on $\mathrm{G}(\mathbf{A})$.

Lemma 4.1.1. — *Let π be an ordinary cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$ of weight $\underline{w}$ over a finite extension L of $\mathbf{Q}_p^{\mathrm{ab}}$. There exists a bilinear pairing*

$$\langle\,,\,\rangle\colon \pi^{\mathrm{ord}} \otimes_L N_{\underline{w}^\vee}(L) \to L$$

such that for all $\iota\colon L \hookrightarrow \mathbf{C}$ extending $\iota^{\mathrm{ab}} = (4.0.3)$ and all sufficiently large $r \in \mathbf{N}^{S_p}$,

$$\iota\langle f, g\rangle = \frac{|D_F|^{1/2}\zeta_F(2)}{\omega_{\pi,p}(-1)e_p(\mathrm{ad}(V_\pi), \psi_p)\cdot 2^{\sum_\sigma -1-w_\sigma}\cdot L(1, \pi^\iota, \mathrm{ad})} q_{F,p}^r(w_{r,p}\mathrm{U}_p^{-r}f^{\iota,\psi,\mathrm{a}}, g^{\iota,\psi}).$$

The pairing $\langle\,,\,\rangle$ satisfies the following properties.

1. *For all $f \in \pi^{\mathrm{ord}}$, $g \in N_{\underline{w}^\vee}(L)$, we have $\langle f, g\rangle = \langle f, e^{\mathrm{ord}}e^{\mathrm{hol}}g\rangle$;*
2. *If $f_0 \in \pi^\infty$, $f_0^\vee \in \pi^{\vee,\infty}$ are ordinary forms, new at places away from p, holomorphic at the infinite places, and with first Fourier coefficients equal to 1, then*

$$\langle f_0, f_0^\vee\rangle = c_{\pi^\infty} \tag{4.1.1}$$

for some constant $c_{\pi^\infty} \in L^\times$ depending only on the Bernstein components and the monodromy of π_v for all $v \nmid p\infty$.

Proof. — The existence and (4.1.1) follow from the factorization of the Petersson inner product into parings in the Whittaker models [CaiST14, Proposition 2.1], together with the local calculations of [CaiST14, Proposition 3.11] away from p and [Dis/b, Lemma A.3.3] at p (the latter needs to be corrected by a factor $\omega_{\pi,p}(-1)$). Since forms in π^{ord} are anti-holomorphic, it is clear that the pairing factors through e^{hol}. ☐

Proposition 4.1.2. — *Let $\mathscr{X}_{\mathrm{G}} \subset \mathscr{Y}_{\mathrm{G}}$ be a Hida family, with sheaf of modular forms Π. There is a unique $\mathscr{O}(\mathscr{Y}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})$-bilinear pairing*

$$\langle\langle\ ,\ \rangle\rangle \colon \Pi \otimes_{\mathscr{O}(\mathscr{Y}_{\mathrm{G}})} \mathscr{S}^{\curlyvee}(\mathscr{Y}_{\mathrm{G}})_{\mathbf{Q}_p^{\mathrm{ab}}} \to \mathscr{K}(\mathscr{X}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})$$

such that for all $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$, corresponding to an ordinary representation $\pi = \pi_x$ over $\mathbf{Q}_p^{\mathrm{ab}}(x)$, and for all $\mathbf{f} \in \Pi_{\mathbf{Q}_p^{\mathrm{ab}}}$, $\mathbf{g} \in \mathscr{S}^{\curlyvee}(\mathscr{Y}_{\mathrm{G}})_{\mathbf{Q}_p^{\mathrm{ab}}}$, we have

$$\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle(x) = \langle \mathbf{f}_x, \mathbf{g}_x\rangle.$$

Proof. — The construction is very similar to that of the pairing denoted by $H^{-1}l_\lambda$ in [Hid91, p. 380]. In this case, let $\mathbf{f}_0^{\curlyvee}$ be the normalized newform in $\Pi^{\curlyvee}$, let $U^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be a maximal open compact subgroup fixing $\mathbf{f}_0^{\curlyvee}$, and let

$$e_{\mathbf{f}_0^{\curlyvee}} \colon \mathscr{M}^{\curlyvee} \to \mathscr{K}(\mathscr{X}_{\mathrm{G}})\, \mathbf{f}_0^{\curlyvee}$$

be the unique $\mathscr{H}^{\mathrm{sph}}_{\mathrm{G}}(U^p)$-equivariant idempotent which factors through the idempotent projection $\mathscr{M}^{\curlyvee} \to \mathscr{M}^{\curlyvee}(U^p)$. Then we define

$$\langle\langle \mathbf{f}_0, \mathbf{g}\rangle\rangle := c_{\mathscr{X}_{\mathrm{G}}}^{-1} \cdot (e_{\mathbf{f}_0^{\curlyvee}}(\mathbf{g}))_1,$$

where the subscript denotes the 1st q-expansion coefficient. Let us explain why the constant $c_{\mathscr{X}_{\mathrm{G}}} := c_{\pi^\infty}$ for $\pi = \Pi_x$ with $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ is well-defined independently of x. As noted before, c_{π^∞} only depends on the Bernstein component and the (rank of the) monodromy of $\pi_{x,v}$ for $v \nmid p\infty$. (In plain terms, the rank of the monodromy is 1 if $\pi_{x,v}$ is a special representation and it is 0 otherwise.) The Bernstein component is an invariant of connected families. As for the rank of the monodromy, by the local-global compatibility result of Proposition 2.4.1, it is the rank of the monodromy of the Weil–Deligne representation attached to $\mathscr{V}_{|x}$. Since the latter is pure, the desired constancy along $\mathscr{X}_{\mathrm{G}}$ follows from [Dis20, Proposition 3.3.1].

In general, we may write $\mathbf{f} = T\mathbf{f}_0$ for some Hecke operator T supported away from p. We then define $\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle := \langle\langle \mathbf{f}_0, T^{\curlyvee}\mathbf{g}\rangle\rangle$. The interpolation property follows from the definitions, the interpolation property proved in [Hid91, Lemma 9.3], and (4.1.1). ☐

4.2. Waldspurger's Rankin–Selberg integral.

— We recall the local and global theory of Waldspurger's [Wal85] integral representation of our L-function.

4.2.1. *Setup.* — Let $\chi \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}}(\mathbf{C})$ be a character of weight $\underline{l}$, let $\kappa_\chi \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ its weight, and let $\omega_\chi := \chi_{|\mathbf{A}_F^\times}$.

Let π be an ordinary complex automorphic representation of $\mathrm{G}(\mathbf{A})$ of weight $\underline{w}$ and central character ω_π, corresponding to a point of $x \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}$ together with an embedding $\iota \colon \mathbf{Q}_p(x) \hookrightarrow \mathbf{C}$. Let $\underline{\kappa}_\pi \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ be the weight of π. Let $\alpha \colon F_p^\times \to \mathbf{C}^\times$ be the character such that $\mathrm{U}_y f = \alpha(y) f$ for any $f \in \pi^{\mathrm{ord}}$ and $y \in F_p^\times$. Let $\underline{\kappa}_\pi = \underline{\kappa}_x \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ be the weight of π. Then

$$\kappa_{\pi,0}(z) = \omega(z) z^{w}, \qquad \kappa_\pi(y) = \alpha_{|U^\circ_{F,p}}(y) y^{(w+w_0)/2}$$

are the decomposition into a product of a smooth and an algebraic character.

Define, as in Corollary 3.4.5, a numerical weight $\underline{k}$ and a smooth character κ_2' of $U^p_{F,0}$ by

$$\begin{aligned} \kappa'_\chi \kappa'_2 = \kappa'_{\pi^\vee} &= \alpha_{|U^\circ_{F,p}}, & \xi &= \omega_\pi \omega_\chi, \\ k &= (w - 2 - |l| - k_0)/2, & k_0 &= w_0 + l_0, \end{aligned} \tag{4.2.1}$$

and let $\underline{\kappa}_2 \in \mathfrak{W}(\mathbf{C})$ be the associated weight as in (3.4.6).

For $v|p$, we choose a Schwartz function $\phi_v \in \mathcal{S}(\mathbf{V}_v \times F_v^\times)$ as in (3.4.1) and (3.4.3) (for ϕ_1), and (3.3.10) and (4.2.1) (for ϕ_2); then

$$\phi_v(x,u) = \delta_{r,v}(x_1)\mathbf{1}_{\mathbf{V}_{2,v}}(x_2)\delta_{U_{F,v}^\circ}(u)\alpha_v(u). \tag{4.2.2}$$

For $v|\infty$, let $\Phi_{l_0,l,k_0,k,v}$ be a preimage, under the map (3.1.1), of

$$\phi_{l_0,l,k_0,k,v} = \phi_{1,l_0,l,v}(x_1,u)\phi_{2,k_0,k,v}(x_2,u),$$

where the factors are defined in (3.2.3), (3.3.2)

4.2.2. *Waldspurger's integral.* — The next proposition gives an integral representation for the L-function we are interested in. We first define the local terms.

For $\Phi \in \mathcal{S}(\mathbf{V}\times \mathbf{A}_F^\times)$, let

$$R_{r,v}(W_v,\Phi_v,\chi_v,\xi_v) = \int_{Z(F_v)N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{-1,v}(g)\delta_{\xi_v,r}(g)\int_{T(F_v)} \chi_v(t) r(gw_{r,v}^{-1})\Phi_v(t^{-1},q(t))\,dt\,dg.$$

Note that the integral $R_{r,v}$ does not depend on $r \geq \underline{1}$ unless $v|p$ and it does not depend on χ if $v|\infty$; we will accordingly simplify the notation in these cases. We also define normalised versions. For $v|p$, let $\Phi_v = \phi_v = (4.2.2)$. Then

(4.2.3)

$$R_v^\natural(W_v,\Phi_v,\chi_v,\xi_v,(k),\psi_v) := \begin{cases} |d_v|^{-2}|D_v|^{-1/2}\dfrac{\zeta_{F,v}(2)L(1,\eta_v\xi_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)} R_v(W_v,\Phi_v,\chi_v,\xi_v,\psi_v) & \text{if } v\nmid p\infty \\ \dfrac{\zeta_{F,v}(2)L(1,\eta_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)} R_v(W_v,\Phi_{l_0,l,k_0,k,v},\chi_v,\xi_v,\psi_v) & \text{if } v|\infty \end{cases}$$

for $v\nmid p$, and

(4.2.4)

$$R_{r,v}^\dagger(W_v,\Phi_v,\chi_v,\xi_v,\alpha_v,\psi_v) := |d_v|^{-2}|D_v|^{-1/2}\frac{\zeta_{F,v}(2)L(1,\eta_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)} q_{F,v}^r\alpha_v^{-r}R_{r,v}(W_v,\Phi_v,\chi_v,\xi_v,\psi_v).$$

for $v|p$.

Proposition 4.2.1. — *Let $f \in \pi^{\mathrm{ord}}$ be antiholomorphic and with factorisable ψ^{-1}-Whittaker function W, and let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty}\times \mathbf{A}^{p\infty,\times})$. For sufficiently large $r=(r_v)_{v|p}$, we have*

$$\iota q_{F,p}^r\alpha(\varpi_p)^{-r}(f, w_{r,p}^{-1}I(\Phi^{p\infty};\chi,\xi,\kappa_2',k)) = |D_F|^{-1}|D_E|^{-1/2}\frac{L(1/2,\pi_E\otimes\chi)}{\zeta_F(2)L(1,\eta)} \cdot\prod_{v\nmid p} R_{r,v}^\natural(W_v,\Phi_v,\chi_v,\xi_v)\prod_{v|p}R_{r,v}^\dagger(W_v,\chi_v,\xi_v,\alpha_v,\psi_v),$$

where all but finitely many of the factors in the infinite product are equal to 1.

Proof. — As in [Dis17, Proof of Proposition 3.5.1], corrected in [Dis/a, Appendix B, under "*Proposition 2.4.4.1*"] to include the factor $q_{F,p}^r$. □

4.2.3. *Non-vanishing of the local integrals.* — We recall a fundamental non-vanishing result for our zeta integrals for selfdual $\pi\boxtimes\chi$, as well as a useful refinement.

Lemma 4.2.2. — *Let $v\nmid p\infty$ be a place of F and let L be a field of characteristic zero. Let π_v be a smooth irreducible representation of $\mathrm{GL}_2(F_v)$ over L, with central character ω_v, and let $\chi_v\colon E_v^\times\to L^\times$ be a smooth character. Assume the self-duality condition $\omega_v\chi_{|F_v^\times} = \mathbf{1}$.*

There exist

- *a 4-dimensional quadratic space $\mathbf{V}_v = \mathbf{B}_v$ over F_v of the type described in § 3.1.4, uniquely determined by*

$$\varepsilon(\mathbf{B}_v) = \eta_v\chi_v(-1)\varepsilon(\pi_{E,v}\otimes\chi_v),$$

– *a function W_v in the Whittaker model of π_v,*
– *a Schwartz function $\phi_v\in\mathcal{S}(\mathbf{V}_v\times F_v^\times,L)$,*

such that

$$R_v(W_v,\phi_v,\chi_v,\psi_v)\neq 0.$$

for every nontrivial character ψ_v.

If moreover all the data are unramified at a place v inert in E, it is possible to choose W_v and $\phi_v=\phi_{1,v}\phi_{2,v}$ such that $\phi_{2,v}(0,u)=0$ for all u (condition (3.3.6)*).*

Proof. — The argument in [Dis17, Proof of Proposition 3.7.1, second paragraph] applies verbatim to prove the first statement. Let us prove the second one. We drop all subscripts v. Fix an isomorphism $\mathbf{V}_2\cong E$, and let us choose W to be a new vector vector, $\phi_{1,v}$ to be the standard Schwartz function, and

$$\phi_2(x_2,u)=\mathbf{1}_{\mathscr{O}_E^\times}(x_2)\mathbf{1}_{\mathscr{O}_F^\times}(u).$$

Writing $\doteq$ for an equality up to nonzero scalars, by the Iwasawa decomposition

$$R(W,\phi,\chi)\doteq\int_{F^\times}W\left(\begin{pmatrix}y&\\&1\end{pmatrix}\right)\int_{E^\times}\chi(t)\int_{\mathrm{GL}_2(\mathscr{O}_F)}r(g)\phi(t^{-1}y,y^{-1}q(t))\,dg\,d^\times y\,dt.$$

Let $U_0(\varpi^r)\subset U_0:=\mathrm{GL}_2(\mathscr{O}_F)$ be the set of matrices which are upper-triangular modulo ϖ^r. It is easy to verify that ϕ_2 is invariant under $U_0(\varpi^r)$ for some r, and that $U_0=U_0(\varpi^r)\sqcup\bigsqcup_{b\in\mathscr{O}_{F,v}/\varpi^t}\left(\begin{smallmatrix}1&\\b&1\end{smallmatrix}\right)U_0(\varpi)$. Thus the integral in dg is a constant multiple of

$$\begin{aligned}&\int_{\mathscr{O}_F}r(w)[\psi(y^{-1}bq(tx'))\hat{\phi}(x',y^{-1}q(t))]_{|x=t^{-1}y}\,db\\
=&\int_{\mathscr{O}_F}\int_{E\times E}\psi(\mathrm{Tr}_{E/F}tx_1)\psi(y^{-1}q(t)\cdot bq(x))\mathbf{1}_{\mathscr{O}_E}(x_1)\widehat{\mathbf{1}_{\mathscr{O}_E^\times}}(x_2)\mathbf{1}_{\mathscr{O}_F}^\times(y^{-1}q(t))]\,dxdb\\
=&\,\mathbf{1}_{\mathscr{O}_F}^\times(y^{-1}q(t))\int_{\mathscr{O}_F}\widehat{\mathbf{1}_{\mathscr{O}_E}}(t)\int_E\psi(y^{-1}q(t)bq(x_2)\widehat{\mathbf{1}_{\mathscr{O}_E^\times}}(x_2)\,dx_2\,db\doteq\mathbf{1}_{\mathscr{O}_E}(t)\mathbf{1}_{\mathscr{O}_F}^\times(y^{-1}q(t))\end{aligned}$$

where the last equality follows from interchanging the order of integration and observing that $\widehat{\mathbf{1}_{\mathscr{O}_E^\times}}(x_2)=\mathrm{vol}(\mathscr{O}_E^\times)$ for $x_2\in\mathscr{O}_E$.

The last quantity equals $\phi^\circ(t^{-1}y,y^{-1}q(t))$ for the standard Schwartz function ϕ°; therefore the integral R is a constant multiple of the unramified integral, in particular it is nonzero. ☐

4.3. Evaluation of the integrals at p and ∞.

— We explicitly compute the local integrals at the places $v|p\infty$.

4.3.1. *p-adic integrals.* — Define, for $v|p$,

$$\begin{aligned}e_v(V_{\pi_E\otimes\chi},\psi_v)&:=\frac{L(1/2,\pi_{E,v}\otimes\chi_v)}{\zeta_{F,v}(2)L(1,\eta_v)})\prod_{w|v}\gamma(\chi_w\alpha_{\pi,v}|\cdot|\circ N_{E_w/F_v},\psi_v)^{-1}\\
e_p(V_{\pi_E\otimes\chi},\psi_p)&:=\prod_{v|p}e_v(V_{\pi_E\otimes\chi},\psi_v).\end{aligned}\tag{4.3.1}$$

Lemma 4.3.1. — *Let $v|p$. Then for any sufficiently large r (depending on χ_v, π_v) we have*

$$R_{r,v}^\dagger(W_v,\phi_v\chi_v,\psi_v)=\frac{\chi_v(-1)}{L(1,\eta_v)}e_v(V_{\pi_E\otimes\chi},\psi_v)$$

Proof. — By [Dis17, Lemma A.2.2] (with the discriminant factors corrected as in [Dis/a, Appendix B]), we have

$$R_{r,v}^\dagger(W_v,\phi_v\chi_v,\psi_v)=\frac{L(1/2,\pi_E\otimes\chi_v)}{\zeta_{F,v}(2)L(1,\eta_v)^2}Z_v,$$

where Z_v are integrals defined in [Dis17, Lemma A.1.1]. By [Dis/a, Lemma A.1.1], we have

$$Z_v = \chi_v(-1)\prod_{w|v}\gamma(\chi_w\alpha_v|\cdot|\circ N_{E_w/F_v},\psi_v)^{-1}.$$

The asserted formula follows. □

4.3.2. *Archimedean integrals.* — The *standard antiholomorphic Whittaker function* for $-\psi$ of weight (w_0, w) is

$$W^{(w_0,w),\mathrm{a}}\left(\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right) r_\theta\right) = z^{w_0}\mathbf{1}_{\mathbf{R}^+}(y)|y|^{(w+w_0)/2}\psi(-x+iy)\psi(-w\theta) \tag{4.3.2}$$

We compute the local integrals R_v when $v|\infty$.

Lemma 4.3.2. — *Let $v|\infty$, let W_v be the standard antiholomorphic Whittaker function of weight (w_0, w) for $\overline{\psi}$, and let $\Phi_v = \Phi_{l_0,l,k_0,k,v}$. Then*

$$R_v^\natural(W_v, \Phi_v, \chi'_v) = i^{-k_0}2^{-1-w}.$$

Proof. — By the Iwasawa decomposition we can uniquely write any $g \in \mathrm{GL}_2(\mathbf{R})$ as

$$g = \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} z & \\ & z\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$

with $x \in \mathbf{R}$, $z \in \mathbf{R}^\times$, $y \in \mathbf{R}^\times$, $\theta \in \mathbf{R}/2\pi\mathbf{Z}$); the local Tamagawa measure is then $dg = dxd^\times z\frac{d^\times y}{|y|}\frac{d\theta}{2}$. Dropping all subscripts v, since the weights match the integration over $\mathrm{SO}(2,\mathbf{R})$ yields 1, and we have

$$R = R(W,\Phi,\chi) = \int_{\mathbf{R}^\times\times(\mathbf{R}^\times\backslash\mathbf{C}^\times)\times\mathbf{R}/2\pi\mathbf{Z}\times\mathbf{R}^\times}\chi(tz)\omega_\pi(z)|y|^{(w+w_0)/2}e^{-2\pi y}\xi^{-1}(z)|y|\Phi(yzt^{-1},y^{-1}z^{-2}q(t)) \, d^\times z\frac{d\theta}{2}\frac{d^\times y}{|y|}\,dt$$

By definition, $\omega_\pi\chi\xi^{-1}(z) = 1$, so that the integration in $d^\times z$ simply realises the map $\Phi \mapsto \phi$. Then

$$\begin{aligned} R &= \pi\int_{\mathbf{R}^\times}\int_{\mathbf{R}^\times\backslash\mathbf{C}^\times}\chi(t)|y|^{(w+w_0)/2}e^{-2\pi y}|y|\mathbf{1}_{\mathbf{R}^+}(y)P_k(0)y^{(|l|+l_0)/2}\chi(t)^{-1}e^{-2\pi y}\frac{d^\times y}{|y|}\,dt \\ &= 2\pi P_{k_0,k}(0)\int_{\mathbf{R}^+}y^{(w+|l|+w_0+l_0)/2}e^{-4\pi y}d^\times y. \end{aligned} \tag{4.3.3}$$

where $2 = \mathrm{vol}(\mathbf{R}^\times\backslash\mathbf{C}^\times)$.

Recall from (4.2.1) and (3.3.1) that $k_0 = w_0 + l_0$, $k = (w-2-|l|-k_0)/2$ and that $P_{k_0,k}(0) = (2\pi i)^{-k_0}(4\pi)^{-k}(k+k_0)!$. Then after a change of variables we have

$$\begin{aligned} R &= (2\pi)^{1-k_0}i^{-k_0}(4\pi)^{-(w-|l|-k_0-2)/2}\Gamma\left(\frac{w-|l|+k_0}{2}\right)(4\pi)^{-(w+|l|+k_0)/2}\Gamma\left(\frac{w+|l|+k_0}{2}\right) \\ &= i^{-k_0}2^{-1-w}\pi^2\Gamma_{\mathbf{C}}\left(\frac{w-l+k_0}{2}\right)\Gamma_{\mathbf{C}}\left(\frac{w+l+k_0}{2}\right). \end{aligned}$$

Now the result follows from identifying

$$\pi^2\Gamma_{\mathbf{C}}\left(\frac{w-l+k_0}{2}\right)\Gamma_{\mathbf{C}}\left(\frac{w+l+k_0}{2}\right) = \frac{L(1,\pi_{E,v}\otimes\chi)}{\zeta_{F,v}(2)L(1,\eta_v)}.$$

□

4.4. Interpolation of the local zeta integral. — Let $\mathscr{X} = \mathscr{X}_{\mathrm{G}}\hat{\times}\mathscr{X}_{\mathrm{H}}$ be a Hida family for $\mathrm{G}\times\mathrm{H}$, and let $\Pi = \Pi_{\mathscr{X}_{\mathrm{G}}}$. Recall the local-global description of Π from § 2.4.7. Let $\mathscr{W}_v$ be the ψ_v^{-1}-Whittaker model of $\Pi_{v,\mathbf{Q}_p^{\mathrm{ab}}}$, which exists since Π_v is co-Whittaker (see [Dis20, § 4.2]). The space $\mathscr{W}_v$ is, as usual, a space of functions on $\mathrm{GL}_2(F_v)$, ψ_v^{-1}-invariant under the action of the unipotent subgroup $N(F_v)$.

Proposition 4.4.1. — *Let $v \nmid p\infty$. There exists an $\mathscr{O}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$-linear map*

$$\mathbf{R}_v \colon \mathscr{W}_v \otimes_{\mathscr{O}(\mathscr{X}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})} \mathscr{O}(\mathscr{X}) \otimes_{\mathbf{Q}_p} \mathcal{S}(\mathbf{V}_v \times F_v^\times) \to \mathscr{O}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$$

such that for all $\mathbf{W}_v \in \mathscr{W}_v$, all $\phi_v \in \mathcal{S}(\mathbf{V}_v \times F_v^\times)$, and all $(x,y) \in \mathscr{X}^{\mathrm{cl}}_{\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C})$, with underlying embedding $\iota\colon \mathbf{Q}_p^{\mathrm{ab}}(x,y) \hookrightarrow \mathbf{C}$, we have

$$\mathbf{R}_v(\mathbf{W}_v, \phi_v)(x,y) = R_v^\natural(\iota\mathbf{W}_{v,x}, \iota\phi_v, \chi^\iota_{y,v}, \psi_v).$$

Proof. — In fact, we may prove a stronger statement by replacing $\mathscr{X}_{\mathrm{H}}$ by (its image in) $\mathscr{Y}_v = \operatorname{Spec} \mathbf{Q}_p[E_v^\times]$, the space of characters of $E_v^\times$, or equivalently any connected component $\mathscr{Y}_v^\circ$ thereof (which is an étale G_m-torsor for $\mathbf{G}_{m,\mathbf{Q}_p}^{w|v}$, the action being induced by multiplication by the uniformisers in $E_v^\times = \prod_{w|v} E_w^\times$).

The proof is largely similar to that of [Dis20, Lemma 5.2.3]; we refer to *loc. cit.* and the sections preceding it for more details on the notions we use. Since $\mathscr{W}_v = \Pi(\mathscr{V}_{\mathscr{X}_{\mathrm{G}},v})$ is in the image of the local Langlands correspondence, there exists an irreducible component $\mathfrak{X}^\circ$ of the extended Bernstein variety of [Dis20, § 3.3] and a map $\mathscr{X}_{\mathrm{G}} \to \mathfrak{X}^\circ$, such that $\mathscr{W}_v$ is a quotient of the universal co-Whittaker module over $\mathfrak{X}^\circ$. We may further extend scalars to $\mathbf{C}$ and replace $\mathfrak{X}^\circ$ by a cover of the form

$$\widetilde{\mathfrak{X}}^\circ = \mathbf{G}_m^d$$

for $d = 1$ or 2; then the pull-back $\widetilde{\mathscr{W}_v}$ of the universal co-Whittaker module has one of the following shapes:

(a) $\widetilde{\mathscr{W}_v} = \operatorname{Ind}_{P_v}^{\mathrm{GL}_2(F_v)}(\alpha_1 \boxtimes \alpha_2)$, where $d = 2$ and $\beta_i \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ are the universal characters;
(b) $\widetilde{\mathscr{W}_v} = \mathrm{St} \otimes \beta_1$, where $d = 1$ and $\beta_1 \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ is the universal character;
(c) $\widetilde{\mathscr{W}_v} = \pi_0 \otimes \beta_1$ where π_0 is a complex supercuspidal representation of $\mathrm{GL}_2(F_v)$, $d = 1$, and $\beta_1 \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ is the universal character.

In all cases, we need to show that for every $\mathbf{W} \in \widetilde{\mathscr{W}_v}$, there is an element $\mathbf{R}_v(\phi_v, \mathbf{W}) \in \mathscr{O}(\widetilde{\mathfrak{X}}^\circ \times \mathscr{Y}_v \times \Psi_v)$ such that

$$\mathbf{R}_v = L(1/2, \pi_{x,E,v} \otimes \chi_{y,v})^{-1} R_v(\phi_v, \mathbf{W}_x, \chi_v);$$

in other words, that the power series in $X_i^{\pm 1} := \beta_i(\varpi_v)^{\pm 1}$ and $Y_w^{\pm 1} := \chi_{\mathrm{univ}}(\varpi_w)^{\pm 1}$ obtained from the integral defining R_v is a Laurent-polynomial multiple of the inverse of the Laurent polynomial $L(1/2, \pi_{x,E,v} \otimes \chi_{y,v})$. This is proved by the same argument as in [Dis17, Proof of Proposition 3.6.1]: since $\widetilde{\mathscr{W}_v}$ is torsion-free, it embeds in the representation $\widetilde{\mathscr{W}_v} \otimes \mathscr{K}(\widetilde{\mathfrak{X}}^\circ)$ over the field $\mathscr{K}(\widetilde{\mathfrak{X}}^\circ)$, so that the usual explicit description of the Kirillov model used in *loc. cit.* applies. □

4.5. The p-adic L-function. — Let

$$\mathscr{X} \subset \mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}}$$

be a Hida family with $\mathscr{X}^{\mathrm{sd}} \neq \emptyset$.

If $\mathscr{X}'$ is an (ind-)scheme over $\mathbf{Q}_p^{\mathrm{ab}}$, we define

$$\mathscr{X}'_{/\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C}) \subset \mathscr{X}'(\mathbf{C})$$

to be the subset of geometric points over ι^{ab} (that is, those such that the composition $\operatorname{Spec} \mathbf{C} \to \mathscr{X}'_{\mathbf{Q}_p^{\mathrm{ab}}} \to \operatorname{Spec} \mathbf{Q}_p^{\mathrm{ab}}$) is $\iota^{\mathrm{ab},\sharp}$).

4.5.1. *Definition and interpolation property.* — Let S be a finite set of places of F, disjoint from $S_{p\infty}$ and containing all those at which the level of $\mathscr{X}$ is not maximal. We fix the isomorphism

$$\mathbf{W} \colon \Pi_{\mathbf{Q}_p^{\mathrm{ab}}} \to \prod_v \mathscr{W}_v$$

$$\mathbf{f} \mapsto \mathbf{W}_{\mathbf{f}}$$

over the open subset of $\mathscr{X}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$ given by Proposition 2.4.1, such that for all classical points $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$ and $\iota\colon \mathbf{Q}_p^{\mathrm{ab}}(x) \hookrightarrow \mathbf{C}$ extending $\iota^{\mathrm{ab}} = (4.0.3)$, we have

$$\mathbf{W}_{\mathbf{f},x}(y_S) = W^{(p)}_{\mathbf{f}_x^{\iota}, y_S 1^{S\infty}}$$

in terms of the setup of § 2.3.2.

For each classical point $(x, y) \in \mathscr{X}^{\mathrm{cl,sd}}$, the application of Lemma 4.2.2 to π_x and χ_y and to all places $v \nmid p\infty$ provides a quadratic space $\mathbf{V}_{(x,y)}$ as in § 3.1.4. By (3.1.3) and the constancy results for epsilon factors in [Dis20, Corollary 5.3.3], the space $\mathbf{V}_{(x,y)} = \mathbf{V}$ is independent of the choice of $(x, y) \in \mathscr{X}^{\mathrm{cl,sd}}$. We fix this space $\mathbf{V}$.

Consider now the restriction to $\mathscr{X}$ of the $(\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}})$-adic form $\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}, x, y)$ of (3.4.10), where $\phi^{p\infty}$ is a Schwartz function on $\mathbf{V}^{p\infty} \times \mathbf{A}_F^{p\infty,\times}$ satisfying (3.3.6) at some place $v \nmid p\infty$ inert in E.

We define, away from the poles of the right-hand side (which depend on $\phi^{p\infty}$), a function

$$\mathscr{L}_p(\mathscr{V})(\phi^{p\infty}) \in \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$$

$$\mathscr{L}_p(\mathscr{V})(\phi^{p\infty})(x, y) := C \frac{\langle\langle \mathbf{f}(x), \mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}; x, y)\rangle\rangle}{\prod_{v\nmid p\infty} \mathbf{R}_v(\mathbf{W}_{\mathbf{f}|x,v}, \phi_v, \chi_{y,v}, \psi_v)}, \tag{4.5.1}$$

where

$$C(x, y) := \omega_x^{p\infty} \omega_y^{p\infty}(-1) L(1, \eta_p) \frac{|D_F|^{-1/2} \zeta_F(2)}{\pi^{[F:\mathbf{Q}]}}.$$

is a constant in $\mathbf{Q}^{\times}$; here, ω_x is the central character of π_x and $\omega_y = \omega_{\chi_y}$. Note that the (base-change of the) functional $\langle\langle \mathbf{f}, -\rangle\rangle$ may be applied to $\mathbf{I}^{\mathrm{ord}}$, thanks to Lemma 2.4.2.

Proposition 4.5.1. — *Let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}_F^{p\infty,\times}$ range among Schwartz functions satisfying that* (3.3.6) *holds at an inert place $v \nmid p\infty$, and that the denominator of* (4.5.1) *is not identically zero. The collection of functions $\mathscr{L}_p(\mathscr{V}, \phi^{p\infty})$ has the following properties.*

1. *Let $(x, y) \in \mathscr{X}^{\mathrm{cl}}_{\mathbf{Q}_p^{\mathrm{ab}}/\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C})$ have contracted weight (k_0, w, l) satisfying*

$$|l_\tau| \leq w_\tau - 2, \qquad |k_0| \leq w_\tau - 2 - |l_\tau|. \tag{4.5.2}$$

   If (x, y) is outside the polar locus of $\mathscr{L}_p(\mathscr{V}, \phi^{p\infty})$, we have

$$\mathscr{L}_p(\mathscr{V})(\phi^{p\infty})(x, y) = e_{p\infty}(V_{(\pi,\chi)} \cdot \mathscr{L}(V_{(\pi,\chi)}, 0), \tag{4.5.3}$$

   where $\pi = \pi_x$, $\chi = \chi_y$.

2. *For each $(x, y) \in \mathscr{X}^{\mathrm{cl,sd}}$, there is a $\phi^{p\infty}$ such that $\mathscr{L}_p(\mathscr{V})(\phi^{p\infty})$ does not have a pole at (x, y).*

Proof. — The third statement follows from Lemma 4.2.2. The second statement follows from the first one, the definition of the factors $e_{p\infty}$, and (4.0.1).

It remains to prove the interpolation property. We have

(4.5.4)

$$\begin{aligned}
\mathscr{L}_p(\mathscr{V})(x, y) &= C \cdot \frac{\iota\langle \mathbf{f}_{x_0}, e^{\mathrm{ord}} \mathrm{I}(\phi^{p\infty}; \chi_{y_0}, \xi_{(x_0,y_0)}, k_{(x_0,y_0)}\rangle}{\prod_{v\nmid p\infty} R_v^{\natural}(\iota W_v, \Phi_v, \chi_{y,v}, \xi_v)} \\
&= C \cdot \frac{|D_F|^{1/2} \zeta_F(2) \cdot |D_F||D_E|^{1/2}}{\omega_{x,p}(-1) \cdot e_p(\mathrm{ad}(V_{\pi_x})(1), \psi_p) \cdot 2^{\sum_{v|\infty} -1-w_v}} \cdot \frac{q_{F,p}^r \alpha_{\pi_{x,p}}^{-r} (\mathbf{f}_x^{\psi}, w_{r,p}^{-1} I_r^{\psi}(\phi^{p\infty}; \chi_y, \xi_{(x,y)}, \kappa_2', k_{(x,y)})}{L(1, \pi_x, \mathrm{ad}) \cdot \prod_{v\nmid p\infty} R_v^{\natural}(\iota W_v, \Phi_v, \chi_{y,v}, \xi_v)} \\
&= C \cdot \frac{\prod_{v|p} R_{r,v}^{\dagger}(W_v, \chi_v, \xi_v, \alpha_v, \psi_v)}{2^{\sum_\tau -1-w_\tau} \omega_{x,p}(-1) \cdot e_p(\mathrm{ad}(V_\pi)(1), \psi_p)} \cdot \frac{|D_F|^{1/2} L(1/2, \pi_E \otimes \chi)}{L(1, \eta) L(1, \pi, \mathrm{ad})} \prod_{v|\infty} R_v^{\natural}(W_v, \Phi_v, \chi_v, \xi_v) \\
&= C \cdot i^{-k_0[F:\mathbf{Q}]} \omega_{x,p} \omega_{y,p}^{\mathrm{sm}}(-1) \frac{e_p(V_{\pi_E \otimes \chi})}{L(1, \eta_p) \cdot e_p(\mathrm{ad}(V_\pi)(1))} \cdot \frac{|D_F|^{1/2} L(1/2, \pi_E \otimes \chi)}{L(1, \eta) L(1, \pi, \mathrm{ad})}. \\
&= i^{k_0[F:\mathbf{Q}]} \cdot e_p(V_{(\pi,\chi)}) \cdot \mathscr{L}(V_{(\pi,\chi)}, 0),
\end{aligned}$$

where in the intermediate steps, $r \in (\mathbf{Z}_{\geq 1})^{S_p}$ is sufficiently large. In the above, we have used in order:

– the defining property of $\langle\langle\, , \rangle\rangle$ in Proposition 4.1.2, and of $\mathbf{R}_v$ in Proposition 4.4.1;
– the interpolation properties of $\mathbf{I}^{\mathrm{ord}}$ in (3.4.11), and of $\langle\ ,\ \rangle$ in Lemma 4.1.1;
– Waldspurger's integral representation as in Proposition 4.2.1;
– the calculations of local integrals in Lemma 4.3.1 and Lemma 4.3.2.

□

4.5.2. *Rationality and completion of the proof of Theorem A.* — By Proposition 4.5.1 and the density of classical points, the functions $\mathscr{L}_p(\phi^{p\infty}) = (4.5.1) \in \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$ glue to a function

$$\mathscr{L}_p(\mathscr{V}) \in \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$$

which satisfies the required interpolation property, and whose polar locus does not meet the set $\mathscr{X}^{\mathrm{cl,sd}}$. All that is left to show that $\mathscr{L}_p(\mathscr{V})$ descends to $\mathscr{K}(\mathscr{X})$. It will be a consequence of the following.

Proposition 4.5.2. — *Let $\mathscr{X}^{\mathrm{cl},\|} \subset \mathscr{X}^{\mathrm{cl}}$ be the sub-ind-scheme of those (x,y) correspond to a representation $\pi \boxtimes \chi$ whose contracted weight (k_0, w, l) satisfies (4.5.2) and is parallel.*[13] *There is a function*

$$\mathrm{L} \in \mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$$

such that for any $z = (x,y) \in \mathscr{X}^{\mathrm{cl},\|}(\mathbf{C})$ corresponding to a representation $\pi \boxtimes \chi$

$$\mathrm{L}(z) = \iota\mathrm{L}(z_0) = i^{-(1+k_0)[F:\mathbf{Q}]}\gamma(1, \eta^\infty\omega^\infty, \psi^\infty)^{-1}\frac{\zeta_F(2)L(1/2, \pi_E \otimes \chi)}{\pi^{[F:\mathbf{Q}]}L(1,\pi,\mathrm{ad})}. \tag{4.5.5}$$

Here, we denote by ω_π be the central character of π, let $\omega_\chi := \chi_{|\mathbf{A}_F^\times}$, let $\omega = \omega_\pi$, and define $\gamma(s, \omega'^\infty, \psi^\infty) := \prod_{v\nmid\infty}\gamma(s, \omega'_v, \psi_v)$.

Remark 4.5.3. — The construction of this paper gives an alternative proof of this result. However, due to the occurrence of the character additive character ψ in the definition of the form I (via the Weil representation), keeping track of rationality requires some burdensome bookkeeping.

Proof. — This is a consequence of a well-known algebraicity theorem of Shimura [Shi78, Theorem 4.2], applied to the newform in the representation π and the CM form attached to χ, whose central character is $\eta\omega_\chi$. (For the comparison of Shimura's periods and adjoint L-values, see [CaiST14, Proposition 1.11].) □

Corollary 4.5.4. — *The function $\mathscr{L}_p(\mathscr{V})$ belongs to $\mathscr{K}(\mathscr{X}) \subset \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$.*

Proof. — We need to show that

$$\mathscr{L}(\mathscr{V})^\sigma = \mathscr{L}(\mathscr{V}) \tag{4.5.6}$$

for all $\sigma \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ab}}/\mathbf{Q}_p)$. Let $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}^{\mathrm{cl},\|,\mathrm{reg}}$ be the intersection of $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}^{\mathrm{cl},\|}$ with the complement of the polar locus of $\mathscr{L}_p(\mathscr{V})$. Since this set is dense in $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}$, it suffices to show that (4.5.6) holds for the restriction $\mathrm{L}_p(\mathscr{V})$ of $\mathscr{L}_p(\mathscr{V})$ to $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}^{\mathrm{cl},\|,\mathrm{reg}}$; in other words, that $\mathrm{L}_p(\mathscr{V})$ belongs to $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}})$.

By (4.5.3) and (4.5.5),

$$\mathrm{L}_p(\mathscr{V})(z) = \frac{i^{[F:\mathbf{Q}]}\gamma(1,\eta^\infty,\psi^\infty)}{L(1,\eta)} \cdot \frac{\gamma(1,\eta^\infty\omega^\infty,\psi^\infty)}{\gamma(1,\eta^\infty,\psi^\infty)\gamma(1,\omega^\infty,\psi^\infty)} \cdot \frac{1}{\gamma(1,\omega^{p\infty},\psi^{p\infty})^{-1}} \cdot \frac{e_p(V_{(x,y)})}{\gamma(1,\omega_p,\psi_p)^{-1}} \cdot \mathrm{L}(z).$$

We show that all factors belong to $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}})$:

– by the class number formula and standard results on Gauß sums, the ratio $L(1,\eta)/i^{[F:\mathbf{Q}]}\gamma(1,\eta^\infty)$ is rational, as both numerator and denominator are rational multiples of $|D_{E/F}|^{-1/2}$;
– by (4.0.1), the second and fourth ratios are values of functions on $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$, as $e_p(V_{(x,y)})$ is a ratio of inverse gamma factors of characters whose ratio is ω_p;

[13] That is, w_τ is independent of $\tau \in \Sigma_\infty$ and so is l_τ. Without this condition, we may have a slightly weaker result.

- as $\mathscr{X}$ is connected and it contains points z with $\omega_z = \mathbf{1}$, the character $\omega_{z,v}$ is unramified for all $z \in \mathscr{X}$ and all $v \nmid p$; thus for those v, the quantity $\gamma(\omega_v)$ is a ratio of L-values, hence the third factor is also the value of a function in $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\parallel})$;
- finally, $\mathrm{L} \in \mathscr{O}(\mathscr{X}^{\mathrm{cl},\parallel})$ by Proposition 4.5.2.

This completes the proof of the corollary and of Theorem A. □

Appendix A. Reality shows and double-factorial identities

Consider the identity

$$(*) \qquad \sum_{k=0}^{n} \binom{n}{k}(2k-1)!!(2n-2k-1)!! = 2^n n!$$

where we recall that $(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1)$ is the number of perfect matchings (into pairs) of a $2m$-element set. Since $\Gamma(j+1/2) = \frac{(2j-1)!!}{2^j}\sqrt{\pi}$, the identity $(*)$ is equivalent to (3.3.8).

Quick analytic proofs of $(*)$ have appeared in [AA10], [GQ12, Theorem 3]. As we were not able to find a bijective proof in the literature, we give one here. Another bijective proof was communicated to the author by David Callan.

A *reality TV show format* is an algorithm whose inputs are called *players' choices* and whose outputs are called *outcomes* (the set of players is partitioned into two disjoint sets, the *producers* and the *participants*). A format is said to be *bijective* if its set of players' choices is in bijection with its set of outcomes.

We will describe two bijective formats for reality TV shows, with different sets of players' choices but the same set of outcomes. In each case, there are $2n$ participants forming an ordered set of n heterosexual couples;(14) there are two tropical islands, Q and H, and in each case, the outcome is:

- a new matching of the participants into n disjoint couples (which may be homosexual or heterosexual), and
- an assignment of each participant to either island Q or island H, such that
- each person lives in the same island as both their old and their new partner.

Show 1. The producers choose a set of couples, send all their members to island Q, and send all the other participants to island H. Within each island, people mingle until they form new disjoint couples (heterosexual or homosexual) as they wish.

Show 2. The producers pick a permutation $\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}$, then they do the following.
- Initialize: $i = 1$ and the set-variable $C = \emptyset$ (where C is for 'cycle'; to be thought of as the set of couples embarked in the show's boat at a given time);
- Process:
  (a) consider couple i, set $C_{\mathrm{new}} = C_{\mathrm{old}} \cup \{i\}$, and interview couple member p_i where: if $C = \{i\}$, then p_i is the woman; if $C \supsetneq \{i\}$, then p_i is the one that does not yet have a new partner.
  The possible answers to the interview question are 'H' and 'Q'.
  (b) If $j = \sigma(i) \notin C$ and p_i responds H (resp. Q):
  - rematch p_i with the person of opposite (resp. same) sex of couple $j = \sigma(i)$. Set $i_{\mathrm{new}} = j$. Return to (a).
  (c) If $j = \sigma(i) \in C$ and p_i responds H (resp. Q):
  - rematch p_i with the unique non-rematched person of couple j, and send all members of the 'original couples' in C to island H (resp. Q); set $C_{\mathrm{new}} = \emptyset$;

(14)These TV shows, for simplicity or close-mindedness, assume the gender binary.

- if everyone has been rematched, STOP. Else: set $i_{\mathrm{new}} \in \{1, \dots, n\}$ to be the smallest such that neither member of couple i_{new} has been rematched. Return to (a).

Proof of $(*)$. — The number of possible players' choices of Show 1 is the left-hand side of $(*)$. The number of possible players' choices of Show 2 is the right-hand side of $(*)$. But the shows are bijective with same set of outcomes. □

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