THE p-ADIC GROSS–ZAGIER FORMULA ON SHIMURA CURVES, II: NON-SPLIT PRIMES

by

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Abstract. — The formula of the title relates p-adic heights of Heegner points and derivatives of p-adic L-functions. It was originally proved by Perrin-Riou for p-ordinary elliptic curves over the rationals, under the assumption that p splits in the relevant quadratic extension. We remove this assumption, in the more general setting of Hilbert-modular abelian varieties.

Contents

1. Introduction and statement of the main result

The p-adic Gross–Zagier formula of Perrin-Riou relates p-adic heights of Heegner points and derivatives of p-adic L-functions. In its original form [PR87], it concerns (modular) elliptic curves over $\mathbf{Q}$, and it is proved under two main assumptions: first, that the elliptic curve is p-ordinary;

secondly, that p splits in the field E of complex multiplications of the Heegner points. (The formula has applications to both the p-adic and the classical Birch and Swinnerton-Dyer conjecture.)

The first assumption was removed by Kobayashi [Kob13] (see also [BPS]). The purpose of this work is to remove the second assumption.

We work in the context of [I], which will enable us in [Dis/b] to deduce, from the formula presented here, the analogous one for higher-weight (Hilbert-) modular motives, as well as a version in the universal ordinary family with some new applications. Nevertheless, the new idea we introduce is essentially orthogonal to previous innovations, including those of [I] (and in fact it can be applied, at least in principle, to the non-ordinary case as well). For this reason we start in § 1.1 by informally discussing it in the simplest classical case of elliptic curves over $\mathbf{Q}$. The general form of our results is presented in § 1.2.

1.1. The main ideas in a classical context. — Classically, Heegner points on the elliptic curve $A_{/\mathbf{Q}}$ are images of CM points (or divisors) on a modular curve X, under a parametrisation $f\colon X \to A$. More precisely, choosing an imaginary quadratic field E, for each ring class character $\chi\colon \mathrm{Gal}(\overline{E}/E) \to \overline{\mathbf{Q}}^{\times}$, one can construct a point $P(f,\chi) \in A_E(\chi)$, the χ-isotypic part of $A(\overline{E})_{\overline{\mathbf{Q}}}$. The landmark formula of Gross–Zagier [GZ86] relates the height of $P(f,\chi)$ to the derivative $L'(A_E\otimes\chi, 1)$ of the L-function of a twisted base-change of A. The analogous formula in p-adic coefficients[(1)]

$$\langle P(f,\chi), P(f,\chi^{-1})\rangle \doteq \frac{d}{ds}_{|s=0} L_p(A_E, \chi\cdot\chi_{\mathrm{cyc}}^s) \tag{1.1.1}$$

relates cyclotomic derivatives of p-adic L-functions to p-adic height pairings $\langle\ ,\ \rangle$. We outline its proof for p-ordinary elliptic curves.

Review of Perrin-Riou's proof. — Our basic strategy is still Perrin-Riou's variant of the one of Gross–Zagier; we briefly and informally review it, ignoring, for simplicity of exposition, the role of the character χ. Letting φ be the ordinary eigenform attached to A, each side of (1.1.1) is expressed as the image under a functional "p-adic Petersson product with φ" (denoted ℓ_φ) of a certain kernel function (a p-adic modular form). For the left-hand side, the form in question is the generating series

$$Z = \sum_{m\geq 0} \langle P^0, T_m P^0\rangle_X \mathbf{q}^m = \sum_v Z_v, \tag{1.1.2}$$

where $P^0 \in \mathrm{Div}^0(X)$ is a degree-zero modification of the CM point $P\in X$, $\langle\ ,\ \rangle_X$ is a p-adic height pairing on X compatible with the one on A, and the decomposition (1.1.2) into a sum running over all finite places of $\mathbf{Q}$ follows from a general decomposition of the global height pairing into a sum of local ones. More precisely, global height pairings are valued in the completed tensor product $H^\times\backslash H^\times_{\mathbf{A}^\infty}\hat{\otimes}L$ of the finite idèles of the Hilbert class field H of E, and of a suitable finite extension L of $\mathbf{Q}_p$. The series Z_v collects the local pairings at $w|v$, each valued in $H_w^\times\hat{\otimes}L$.

The analytic kernel $\mathscr{I}'$ is the derivative of a p-adic family of mixed theta-Eisenstein series. It also enjoys a decomposition

$$\mathscr{I}' = \sum_{v\neq p} \mathscr{I}'_v$$

where, unlike (1.1.2), the sum runs over the finite places of $\mathbf{Q}$ *different from* p. Once established that $Z_v \doteq \mathscr{I}'_v$ for $v\neq p$ by computations similar to those of Gross–Zagier, it remains to show that the p-adic modular form Z_p is annihilated by ℓ_φ.

(1) We denote by '$\doteq$' equality up to a less important nonzero factor.

In order to achieve this, one aims at showing that, after acting on Z_p by a Hecke operator to replace P^0 by $P^{[\varphi]}$ (a lift of the component of its image in the φ-part of $\mathrm{Jac}(X)$), the resulting form $Z_p^{[\varphi]}$ is *p-critical*. That is, its coefficients

$$a_{mp^s} := \langle P^{[\varphi]}, T_{mp^s} P^0 \rangle_{X,p}$$

decay p-adically no slower than a constant multiple of p^s. The p-shift of Fourier coefficients extends the action on modular forms of the operator U_p – which in contrast acts invertibly on the ordinary form φ: this implies that p-critical forms are annihilated by ℓ_φ.

To study the terms a_{mp^s}, one constructs a sequence of points $P_s \in X_H$ whose fields of definitions are the layers H_s of the anticyclotomic p^∞-extension of E. The relations they satisfy allow to express

$$a_{mp^s} = \langle P^{[\varphi]}, D_{m,s} \rangle_{X,p},$$

where $D_{m,s}$ is a degree-zero divisor supported at Hecke-translates of P_s, which are all essentially CM points of conductor p^s defined over H_s. By a projection formula for $X_{H_s} \to X_H$, the height a_{mp^s} is then a sum, over primes w of H above p, of the images

$$N_{s,w}(h_{m,s,w})$$

of heights $h_{m,s,w}$ computed on $X_{H_{s,w}}$, under the norm map $N_{s,w} \colon H_{s,w}^\times \hat{\otimes} L \to H_w^\times \hat{\otimes} L$. Moreover it can be shown that the L-denominators of $h_{m,s,w} \in H_{s,w}^\times \hat{\otimes} L$ are uniformly bounded, so that here we may ignore them and think of $h_{m,s,w} \in H_{s,w}^\times \hat{\otimes} \mathscr{O}_L$.

A simple observation from [I] is that the valuation $w(h_{m,s,w})$ equals

$$m_{X_{H_w}}(P^{[\varphi]}, D_{m,s}), \tag{1.1.3}$$

the intersection multiplicity of the *flat extensions* (§4.2) of those divisors to some regular integral model $\mathscr{X}$ of X_{H_w}. In the split case, it is almost immediate to see that this intersection multiplicity vanishes. This implies that

$$N_{s,w}(h_{m,s}) \in N_{s,w}(\mathscr{O}_{H_{s,w}}^\times) \hat{\otimes} \mathscr{O}_L \subset H_w^\times \hat{\otimes} \mathscr{O}_L. \tag{1.1.4}$$

Since the extension $H_{s,w}/H_w$ is totally ramified of degree p^s, the subset in (1.1.4) is $p^s(\mathscr{O}_{H_w}^\times \hat{\otimes} \mathscr{O}_L) \subset p^s(H_w^\times \hat{\otimes} \mathscr{O}_L)$, as desired.

The non-split case. — In the non-split case, the p-adic intersection multiplicity has no reason to vanish. However, the above argument will still go through if we more modestly show that (1.1.3) itself decays at least like a multiple of p^s. The idea to prove this is very simple: we show that if s is large then, for the purposes of computing intersection multiplicities with other divisors $\mathscr{D}$ on $\mathscr{X}$, the Zariski closure of a CM point of conductor at least p^s can *almost* be approximated by some irreducible component V of the special fibre of $\mathscr{X}$; hence the multiplicity will be zero if $\mathscr{D}$ arises as a flat extension of its generic fibre. The qualifier 'almost' means that the above holds *except* if $|\mathscr{D}|$ contains V itself, which will be responsible for a multiplicity error term equal to a constant multiple of p^s.

The approximation result is precisely formulated in an (ultra)metric space of irreducible divisors on the local ring of a regular arithmetic surface, which we introduce following a recent work of García Barroso, González Pérez and Popescu-Pampu [GBGPPP18]. The proof of the result is also rather simple (albeit not effective), relying on Gross's theory of quasicanonical liftings [Gro86]. The problem of effectively identifying the approximating divisor V is treated in [Dis/a].

Subtleties. — The above description ignores several difficulties of a relatively more technical nature, most of which we deal with by the representation-theoretic approach of [I] (in turn adapted from Yuan–Zhang–Zhang [YZZ12]). Namely, we allow for arbitrary modular parametrisations f, resulting into an extra parameter ϕ in the kernels Z and $\mathscr{I}'$. By representation-theoretic results, one is free to some extent to *choose* the parameter ϕ to work with without losing generality. A fine choice (or rather a pair of choices) for its p-adic component is dictated by the goal of interpolation, while imposing suitable conditions on its other components allows to circumvent many obstacles in the proof.

1.2. Statement. — We now describe our result in the general context in which we prove it – which is the same as that of [I] (and [YZZ12]), to which we refer for a less terse discussion of the background. (At some points, we find some slightly different formulations or normalisations from those of [I] to be more natural: see §2.2 for the equivalence.)

Abelian varieties parametrised by Shimura curves. — Let F be a totally real field and let $A_{/F}$ be a simple abelian variety of GL_2-type. Assume that $L(A,s)$ is modular (this is known in many case if A is an elliptic curve). Let $\mathbf{B}$ be a quaternion algebra over the adèles $\mathbf{A}=\mathbf{A}_F$, whose ramification set $\Sigma_{\mathbf{B}}$ has odd cardinality and contains all the infinite places. To $\mathbf{B}$ is attached a tower of Shimura curves $(X_{U/F})_{U\subset\mathbf{B}^{\infty\times}}$, with respective Albanese varieties J_U. It carries a canonical system of divisor classes $\xi_U\in\mathrm{Cl}(X_U)_{\mathbf{Q}}$ of degree 1, providing a system ι_ξ of maps $\iota_{\xi,U}\in\mathrm{Hom}\,(X_U,J_U)_{\mathbf{Q}}$ defined by $D\mapsto D-\deg(D)\xi_U$. The space

$$\pi=\pi_{A,\mathbf{B}}=\varinjlim_U \mathrm{Hom}^{\,0}(J_U,A)$$

is either zero or a smooth irreducible representation of $\mathbf{B}^\times$ (trivial at the infinite places), with coefficients in the number field $M:=\mathrm{End}^{\,0}(A)$. We assume we are in the case $\pi=\pi_{A,\mathbf{B}}\neq 0$, which under the modularity assumption and the condition (1.2.2) below can be arranged by suitably choosing $\mathbf{B}$. Then for all places $v\nmid\infty$, $L_v(A,s)=L_v(s-1/2,\pi)$ in $M\otimes\mathbf{C}$. We denote by $\omega\colon F^\times\backslash\mathbf{A}^\times\to M^\times$ the central character of π. We have a canonical isomorphism $\pi_{A^\vee,\mathbf{B}}\cong\pi_{A,\mathbf{B}}^\vee$, see [YZZ12, §1.2.2], and we denote by $(\ ,\ )_\pi\colon\pi_{A,\mathbf{B}}\otimes\pi_{A^\vee,\mathbf{B}}\to M$ the duality pairing.

Heegner points. — Let E/F be a CM quadratic extension with associated quadratic character η, and assume that E admits an $\mathbf{A}$-embedding $E_{\mathbf{A}}\hookrightarrow\mathbf{B}$, which we fix. Then $E^\times$ acts on the right on $X=\varprojlim_U X_U$. The fixed-points subscheme $X^{E^\times}\subset X$ is F-isomorphic to $\mathrm{Spec}\,E^{\mathrm{ab}}$, and we fix a point $P\in X^{E^\times}(E^{\mathrm{ab}})$. Let

$$\chi\colon E^\times\backslash E^\times_{\mathbf{A}^\infty}\cong\mathrm{Gal}(E^{\mathrm{ab}}/E)\to L(\chi)^\times$$

be a character valued in a field extension of M, satisfying

$$\omega\cdot\chi_{|\mathbf{A}^{\infty\times}}=\mathbf{1},$$

and let

$$A_E(\chi):=(A(E^{\mathrm{ab}})\otimes_M L(\chi)_\chi)^{\mathrm{Gal}(E^{\mathrm{ab}}/E)},$$

where $L(\chi)_\chi$ is an $L(\chi)$-line with Galois action by χ.

Then we have a *Heegner point functional*

$$f\quad\mapsto\quad P(f,\chi):=\fint_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} f(\iota_\xi(P)^\tau)\otimes\chi(\tau)\,d\tau\quad\in A_E(\chi)\tag{1.2.1}$$

(integration for the Haar measure of volume 1) in the space of invariant linear functionals

$$\mathrm{H}(\pi_{A,\mathbf{B}},\chi)\otimes_{L(\chi)}A_E(\chi),\qquad \mathrm{H}(\pi,\chi):=\mathrm{Hom}_{\,E^\times_{\mathbf{A}}}(\pi\otimes\chi,L(\chi))$$

where $E^\times_{\mathbf{A}}$ acts diagonally. There is a product decomposition $\mathrm{H}(\pi,\chi)=\bigotimes_v\mathrm{H}(\pi_v,\chi_v)$, where similarly $\mathrm{H}(\pi_v,\chi_v):=\mathrm{Hom}_{\,E_v^\times}(\pi_v\otimes\chi_v,L(\chi))$.

A local unit of measure for invariant functionals. — By foundational local results of Waldspurger, Tunnell, and Saito, the dimension of $\mathrm{H}(\pi,\chi)$ (for any representation π of $\mathbf{B}^\times$) is either 0 or 1. If A is modular and the global root number

$$\varepsilon(A_E\otimes\chi)=-1 \tag{1.2.2}$$

then the set of local root numbers determines a unique quaternion algebra $\mathbf{B}$ over $\mathbf{A}$, satisfying the conditions required above and containing $E_{\mathbf{A}}$, such that $\pi_{A,\mathbf{B}}\neq 0$ and $\dim_{L(\chi)}\mathrm{H}(\pi_{A,\mathbf{B}},\chi)=1$.(2) We place ourselves in this case; then there is a canonical factorisable generator

$$Q_{(,),dt}=\prod_v Q_{(,)_v,dt_v}\in \mathrm{H}(\pi,\chi)\otimes\mathrm{H}(\pi^\vee,\chi^{-1})$$

depending on the choice of a pairing $(,)=\prod_v(,)_v\colon \pi\otimes\pi^\vee\to L(\chi)$ and a measure $dt=\prod_v dt_v$ on $E_{\mathbf{A}}^\times/\mathbf{A}^\times$. It is defined locally as follows. Let us use symbols $V_{(A,\chi)}$ and $V_{(A,\chi),v}$, which we informally think of as denoting (up to abelian factors) the 'virtual motive over F with coefficients in $L(\chi)$'

$$\text{“}V_{(A,\chi)}=\mathrm{Res}_{E/F}(h_1(A_E)\otimes\chi)\ominus \mathrm{ad}(h_1(A)(1))\text{”}$$

and its local components (the associated local Galois representation or, if v is archimedean, Hodge structure). Then we let, for each place v of F and any auxiliary $\iota\colon L(\chi)\hookrightarrow \mathbf{C}$,(3)

$$\mathscr{L}(\iota V_{(A,\chi),v},s):=\frac{\zeta_{F,v}(2)L(1/2+s,\iota\pi_{E,v}\otimes\iota\chi_v)}{L(1,\eta_v)L(1,\iota\pi_v,\mathrm{ad})}\cdot\begin{cases}1 & \text{if } v \text{ is finite}\\ \pi^{-1} & \text{if } v|\infty\end{cases}\in\iota L(\chi), \tag{1.2.3}$$

$$Q_{(,)_v,dt_v}(f_{1,v},f_{2,v},\chi_v):=\iota^{-1}\mathscr{L}(\iota V_{(A,\chi),v},0)^{-1}\int_{E_v^\times/F_v^\times}\chi(t_v)(\pi_v(t_v)f_{1,v},f_{2,v})_v\,dt_v. \tag{1.2.4}$$

We make the situation more canonical by choosing $dt=\prod_v dt_v$ to satisfy

$$\mathrm{vol}(E^\times\backslash E_{\mathbf{A}}^\times/\mathbf{A}^\times,dt)=1$$

and by defining for any $f_3\in\pi$, $f_4\in\pi^\vee$ such that $(f_3,f_4)\neq0$:

$$Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4};\chi\right):=\frac{Q_{(,),dt}(f_1,f_2,\chi)}{(f_3,f_4)}. \tag{1.2.5}$$

p-adic heights. — Let us fix a prime $\mathfrak{p}$ of M and denote by p the underlying rational prime. Suppose from now on that for each $v|p$, A_{F_v} has potentially $\mathfrak{p}$-ordinary (good or semistable) reduction. That is, that for a sufficiently large finite extension $L\supset M_{\mathfrak{p}}$, the rational $\mathfrak{p}$-Tate module $W_v:=V_pA\otimes_M L$ is a reducible 2-dimensional representation of $\mathrm{Gal}(\overline{F}_v/F_v)$:

$$0\to W_v^+\to W_v\to W_v^-\to 0. \tag{1.2.6}$$

Fix such a coefficient field L, and for each $v|p$ let $\alpha_v\colon F_v^\times\cong \mathrm{Gal}(F_v^{\mathrm{ab}}/F_v)\to L^\times$ be the character giving the action on the twist $W_v^+(-1)$. The field $L(\chi)$ considered above will from now on be assumed to be an extension of L. Under those conditions there is a canonical p-adic *height pairing*

$$\langle\ ,\ \rangle\colon A_E(\chi)\otimes A_E^\vee(\chi^{-1})\to\Gamma_F\hat{\otimes}L(\chi),$$

where $\Gamma_F:=\mathbf{A}^\times/\overline{F^\times\widehat{\mathscr{O}}_F^{p,\times}}$ (the bar denotes Zariski closure). It is normalised 'over F' as in [**I**, §4.1].

For $f_1,f_3\in\pi$, $f_2,f_4\in\pi^\vee$, and $P^\vee\colon\pi^\vee\otimes\chi\to A_E^\vee(\chi^{-1})$ the Heegner point functional of the dual, our result will measure the ratio $\langle P(f_1,\chi),P^\vee(f_2,\chi^{-1})\rangle/(f_3,f_4)_\pi$, against the value at the

(2) If $\varepsilon(A_E\otimes\chi)=+1$ there is no such quaternion algebra and all Heegner points automatically vanish.

(3) Explicitly, if v is archimedean we have $\mathscr{L}(V_{(A,\chi),v},0)=2$ and $Q_{(,)_v,dt_v}(f_{1,v},f_{2,v})=2^{-1}\mathrm{vol}(\mathbf{C}^\times/\mathbf{R}^\times,dt_v)(f_{1,v},f_{2,v})$.

f_i of the 'unit' Q. The size will be given by the derivative of the p-adic L-function that we now define.

The p-adic L-function. — We continue to assume that A is potentially $\mathfrak{p}$-ordinary, and review the definition of the p-adic L-function from [I, Theorem A] (in an equivalent form). We start by defining the space on which it lives. Write $\Gamma_F = \varprojlim_n \Gamma_{F,n}$ as the limit of an inverse system of finite groups, and let

$$\mathscr{Y}_F^{\text{l.c.}} := \bigcup_n \operatorname{Spec} L[\Gamma_{F,n}] \quad \subset \quad \mathscr{Y}_F := \operatorname{Spec} \mathscr{O}_L[\![\Gamma_F]\!] \otimes_{\mathscr{O}_L} L. \tag{1.2.7}$$

Then $\mathscr{Y}_F$ is a space of continuous characters on Γ_F, and the 0-dimensional ind-scheme $\mathscr{Y}_F^{\text{l.c.}}$ is its subspace of locally constant (finite-order) characters. We denote by $\chi_{F,\text{univ}} \colon \Gamma_F \to \mathscr{O}(\mathscr{Y}_F)^\times$ the tatutological character.

Let

$$\Psi_p := \prod_{v|p} \Psi_v, \qquad \Psi_v := \bigcup_{n\geq 0} \varprojlim_{m\geq 0} \operatorname{Spec} \mathbf{Q}[\varpi_v^{-n}\mathscr{O}_{F,v}/\varpi_v^m \mathscr{O}_{F,v}] - \{\mathbf{1}\}$$

be the ind-scheme of everywhere nontrivial characters of $F_p = F \otimes \mathbf{Q}_p$ (here ϖ_v denotes a uniformizer at v). It is a principal homogeneous space for the locally profinite group scheme $F_p^\times$, which acts by $a.\psi_p(x) = \psi_p(ax)$. For any $\mathbf{Q}$-scheme Y and character $\chi_{F,Y} \colon F_p^\times \to \mathscr{O}(Y)^\times$, denote by $\mathscr{O}(Y \times \Psi_p, \chi_{F,Y}) \subset \mathscr{O}(Y \times \Psi_p)$ the space of functions G satisfying $G(y, a.\psi_p) = \chi_{F,Y}(a) G(y, \psi_p)$ for all $a \in F_p^\times$. We identify $\mathscr{O}(Y \times \Psi_p, \mathbf{1}) = \mathscr{O}(Y)$ via the pullback.

For a character $\chi' \colon \operatorname{Gal}(E^{\text{ab}}/E) \to L'^\times$ together with an embedding $\iota \colon L' \to \mathbf{C}$, consider the ratio of complete L-functions

$$\mathscr{L}(\iota V_{(A,\chi')}, s) := \prod_v \mathscr{L}(\iota V_{(A,\chi'),v}, s), \qquad \mathscr{L}(\iota V_{(A,\chi'),v}, s) = (1.2.3), \qquad \Re(s) \gg 0.$$

Finally, recall that the (inverse) Deligne–Langlands gamma factor of a continuous representation ρ of $\operatorname{Gal}(\overline{F}_v/F_v)$ over a p-adic field L', with respect to a nontrivial character $\psi_v \colon F_v \to \mathbf{C}^\times$ and an embedding $\iota \colon L' \hookrightarrow \mathbf{C}$, is defined as

$$\gamma(\iota\rho, \psi_v)^{-1} := \frac{L(\iota \mathrm{WD}(\rho))}{\varepsilon(\iota \mathrm{WD}(\rho), \psi_v) L(\iota \mathrm{WD}(\rho^*(1)))},$$

where ιWD is the functor to complex Weil–Deligne representations.(4) The function γ extends additively to a function on the Grothendieck group of representations. For $v|p$ and $\chi' \colon \operatorname{Gal}(\overline{E}/E) \to L(\chi')^\times$, define the virtual representation over $L(\chi')$

$$V^+_{(A,\chi'),v} := (W_v^+ \otimes (\operatorname{Ind}_E^F \chi')_v) \ominus \operatorname{ad}(W_v)(1)^{++},$$

where $\operatorname{ad}(W_v)(1)^{++} := \operatorname{Hom}(W_v^-, W_v^+)(1) = \omega_v^{-1}\alpha_v^2 |\ |_v^2$.

Theorem A. — *There is a function*

$$\mathscr{L}_p(V_{(A,\chi)}) \in \mathscr{O}(\mathscr{Y}_F \times \Psi_p, \eta_p \chi^2_{F,\text{univ},p})$$

characterised by the following property. For each complex geometric point $s = (\chi_F, \psi_p) \in \mathscr{Y}_F^{\text{l.c.}} \times \Psi_p(\mathbf{C})$, with underlying embedding $\iota \colon L(\chi_F) \hookrightarrow \mathbf{C}$,

$$\mathscr{L}_p(V_{(A,\chi)}, s) = \iota e_p(V_{(A,\chi')}, \psi_p) \cdot \mathscr{L}(\iota V_{(A,\chi')}, 0), \qquad \chi' := \chi \cdot \chi_{F|\operatorname{Gal}(\overline{E}/E)}.$$

Here the p-correction factor is

$$e_p(V_{(A,\chi')}, \psi_p) := \iota^{-1} \prod_{v|p} \gamma(\iota V^+_{(A,\chi'),v}, \psi_v)^{-1},$$

(4) The terms L and ε are normalised as in [Tat79].

where $\psi_p = \prod_v \psi_v$ is any decomposition.

The factor $e_p(V_{(A,\chi')})$ coincides with the one predicted by Coates and Perrin-Riou (see [Coa91]) for $V_{(A,\chi)}$, up to constants depending only on E and the removal of a trivial zero from their interpolation factor for $\mathrm{ad}(W_v)(1)$.

The p-adic Gross–Zagier formula. — We are almost ready to state our main result. We say that χ_p is *not exceptional* for A if $e_p(V_{(A,\chi)}) \neq 0$. We say that χ_p is *sufficiently ramified* if it is nontrivial on a certain open subgroup of $\mathscr{O}_{E,p}^\times$ depending only on ω_p (see Assumption 3.4.1 below).

Denote by $0 \in \mathscr{Y}_F$ the point corresponding to $\chi_F = \mathbf{1}$, and let

$$\mathscr{L}_p'(V_{(A,\chi)}, 0) := \mathrm{d}\,\mathscr{L}_p(V_{(A,\chi)}, 0) \in T_0\mathscr{Y}_F \cong \Gamma_F \hat{\otimes} L(\chi).$$

Theorem B. — *Suppose that the abelian variety $A_{/F}$ is modular and potentially $\mathfrak{p}$-ordinary. Let $\chi\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to L(\chi)^\times$ be a finite-order character satisfying*

$$\varepsilon(A_E \otimes \chi) = -1,$$

and suppose that χ_p is not exceptional for A and sufficiently ramified.

Then for any $f_1, f_3 \in \pi$, $f_2, f_4 \in \pi^\vee$ such that $(f_3, f_4)_\pi \neq 0$, we have

$$\frac{\langle P(f_1,\chi), P^\vee(f_2,\chi^{-1})\rangle}{(f_3,f_4)_\pi} = e_p(V_{(A,\chi)})^{-1} \cdot \mathscr{L}_p'(V_{(\mathbf{A},\chi)}, 0) \cdot Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}; \chi\right)$$

in $\Gamma_F \hat{\otimes} L(\chi)$.

Remark 1.2.1. — The technical assumption that χ_p is sufficiently ramified is removed by p-adic analytic continuation in [Dis/b]. (Note that for the removal of this assumption, one only needs the anticyclotomic formula analogous to [I, Theorem C.4], and not the more general [Dis/b, Theorem D]; in particular, Theorem B holds for all χ independently of [Dis/b, Conjecture (L_p)].)

The theorem has familiar applications extending to the non-split case those from [I] (when the other ingredients are available); we leave their formulation to the interested reader, and highlight instead an application specific to this case pointed out in [Dis20], as well as a new application to the non-vanishing conjecture for p-adic heights.

A new proof of a result of Greenberg–Stevens. — Note first that concrete versions of the formula of Theorem B may be obtained by choosing explicit parametrisations f_i and evaluating the term Q. This is a local problem, solved in [CST14]. In particular, by starting from Theorem B and applying the same steps of the proofs of [Dis20, Theorems 4.3.1, 4.3.3], we obtain the simple p-adic Gross–Zagier formula in anticyclotomic families for elliptic curves $A_{/\mathbf{Q}}$ proposed in [Dis20, Conjecture 4.3.2].

Now as noted in Remark 5.2.3 *ibid.*, such formula, combined with a result of Bertolini–Darmon, gives yet another proof (quite likely the most complicated so far, but amenable to generalisations) of the following famous result of Greenberg and Stevens [GS93]. If $A_{/\mathbf{Q}}$ is an elliptic curve of split multiplicative reduction at p, with Néron period Ω_A and p-adic L-function $L_p(A,-)$ on $\mathscr{Y}_{\mathbf{Q}}$, then $L_p(A,\mathbf{1}) = 0$ and

$$L_p'(A,\mathbf{1}) = \lambda_p(A) \cdot \frac{L(A,1)}{\Omega_A}, \tag{1.2.8}$$

where $\lambda_p(A)$ is the $\mathcal{L}$-invariant of Mazur–Tate–Teitelbaum [MTT86].

We recall a sketch of the argument, referring to [Dis20] for more details. One chooses an imaginary quadratic field E such that p is inert in E and that the twist $A^{(E)}$ satisfies $L(A^{(E)},1) \neq 0$. By the anticyclotomic p-adic Gross–Zagier formula, $L_p'(A_E,\mathbf{1})$ is the value at $\chi = \mathbf{1}$ of the height of an anticyclotomic family $\mathscr{P}$ of Heegner points. It is shown in [BD01, §5.2] that the value $\mathscr{P}(\mathbf{1})$ equals, in an extended Selmer group, the Tate parameter $q_{A,p}$ of $A_{\mathbf{Q}_p}$ multiplied by a square root of $L(A_E,1)/\Omega_{A_E}$. The height of $q_{A,p}$, in the 'extended' sense of [MTT86, Nek06], essentially equals $\lambda_p(A)$. This shows that, after harmlessly multiplying by $L(A^{(E)},1)/\Omega_{A^{(E)}}$, the two sides of (1.2.8) are equal.

Exceptional cases and non-vanishing results. — Suppose that $A_{/\mathbf{Q}}$ has multiplicative reduction at a prime p inert in E, and that $L(A_E,1)\neq 0$. Then *for all but finitely many anticyclotomic characters χ of p-power conductor, a Heegner point in $A_E(\chi)$ is nonzero and the p-adic height pairing on $A_E(\chi)$ is nondegenerate.* This follows from noting, similarly to the above, that in the p-adic Gross–Zagier formula in anticyclotomic families for A_E, both sides are nonzero since the heights side specialises, at the character $\chi=\mathbf{1}$, to a nonzero multiple of $\lambda_p(A)$, which is in turn nonzero by [BSDGP96].

A similar argument, applied to the formula in Hida families of [Dis/b], will yield the following result: if $A_{/\mathbf{Q}}$ is an elliptic curve with multiplicative reduction and $L(A,1)\neq 0$, then the Selmer group of the selfdual Hida family $\mathbf{f}$ through A has generic rank one, and both the height regulator and the cyclotomic derivative of the p-adic L-function of $\mathbf{f}$ do not vanish. The details will appear in [Dis/b].

1.3. Organisation of the paper. — In §2, we restate our theorems in an equivalent form, direct generalisation of the statements from [I] (up to a correction involving a factor of 2, discussed in Appendix B). In §3 we recall the proof strategy from [I], with suitable modifications and corrections. The new argument to treat p-adic local heights in the non-split case is developed in §4.

We conclude with two appendices, one dedicated to some local results, the other containing a list of errata to [I].

2. Comparison with [I]

We compare Theorems A and B with the corresponding results from [I]. We continue with the setup and notation of §1.2.

2.1. The p-adic L-function. — We deduce our Theorem A from [I, Theorem A].

Let σ^∞ be the nearly $\mathfrak{p}$-ordinary, M-rational ([I, Definition 1.2.1]) representation of $\mathrm{GL}_2(\mathbf{A})$ attached to A as in [I]. In Theorem A *ibid.* we have constructed a p-adic L-function

$$L_{p,\alpha}(\sigma_E),$$

which is a bounded function on the product of a rigid space $\mathscr{Y}'^{\mathrm{rig}}_{/L}$ (denoted by $\mathscr{Y}'$ in [I]) and a scheme Ψ_p° (denoted by Ψ_p in [I]). The space $\mathscr{Y}'^{\mathrm{rig}}=\mathscr{Y}'^{\mathrm{rig}}_{\omega,V^p}$ parametrises certain continuous p-adic characters of $E^\times\backslash E^\times_{\mathbf{A}}$ invariant under an arbitrarily fixed compact open subgroup $V^p\subset E^\times_{\mathbf{A}^{p\infty}}$; and the scheme Ψ_p° parametrises characters of F_v of a fixed level. The boundedness means precisely that we may (and do) identify $L_{p,\alpha}(\sigma_E)$ with a function on a corresponding scheme

$$\mathscr{Y}'\subset \mathrm{Spec}\,\mathscr{O}_L[\![E^\times\backslash E^\times_{\mathbf{A}^\infty}/V^p]\!]\otimes L, \tag{2.1.1}$$

which, when also viewed as a space of characters χ', is the subscheme cut out by the closed condition $\omega\cdot\chi'_{|\widehat{\mathscr{O}}_F^{p,\times}}=\mathbf{1}$. Similarly to $\mathscr{Y}_F$, the scheme $\mathscr{Y}'$ contains a 0-dimensional subscheme $\mathscr{Y}'^{\text{l.c.}}$ parametrising the locally constant characters in $\mathscr{Y}'$. The function $L_{p,\alpha}(\sigma_E)$ is characterised by the following property. Denote by D_K the discriminant of a number field K. Then at all

$(\chi', \psi) \in \mathscr{Y}' \times \Psi_p^\circ$ with underlying embedding $\iota\colon L \hookrightarrow \mathbf{C}$, we have

$$L_{p,\alpha}(\sigma_E)(\chi', \psi_p) = \prod_{v|p} Z_v^\circ(\chi'_v, \psi_v) \cdot \frac{\pi^{2[F:\mathbf{Q}]}|D_F|^{1/2}}{2\zeta_F(2)} \cdot \mathscr{L}(\iota V_{(A,\chi')}) \tag{2.1.2}$$

for certain local factors Z_v°.

Fix a finite-order character

$$\chi\colon E^\times \backslash E^\times_{\mathbf{A}^\infty} \cong \mathrm{Gal}(E^{\mathrm{ab}}/E) \to L(\chi)^\times$$

satisfying $\omega \cdot \chi_{|\mathbf{A}^{\infty\times}} = \mathbf{1}$, and consider the map

$$j_\chi\colon \mathscr{Y}_F \to \mathscr{Y}'$$
$$\chi_F \mapsto \chi \cdot \chi_F \circ N_{E_{\mathbf{A}^{\infty\times}}/\mathbf{A}^{\infty\times}}.$$

Proof of Theorem A. — For $\psi_p = \prod_v \psi_v \in \Psi_p$, let

$$C(\chi'_p, \psi_p) := \frac{e_p(V_{(A,\chi')}, \psi_p)}{\prod_{v|p} Z_v^\circ(\chi'_v, \psi_v)}. \tag{2.1.3}$$

We show in Proposition A.1.4 that this is independent of χ'_p and defines an element $C \in \mathscr{O}(\Psi_p, \omega_p\alpha_p^{-2}|\cdot|_p^{-2})$.

Define

$$\mathscr{L}_p(V_{(A,\chi)}) := \frac{2\zeta_F(2)}{\pi^{2[F:\mathbf{Q}]}|D_F|^{1/2}} \cdot C \cdot L(1, \sigma_v, \mathrm{ad}) \cdot j_\chi^* L_{p,\alpha}(\sigma_E) \tag{2.1.4}$$

a function in $\mathscr{O}(\mathscr{Y}_F \times \Psi_p^\circ)$. By the equivariance properties on Ψ_p° of $C(\cdot)$ from above, and of $L_{p,\alpha}(\sigma_E)$ from [I, Theorem A] as corrected in Appendix B, the function $\mathscr{L}_p(V_{(A,\chi)})$ is a section ot $\mathscr{O}(\mathscr{Y}_F \times \Psi_p^\circ, \chi^2_{F,\mathrm{univ},p})$. Hence we can uniquely extend it to a function on $\mathscr{Y}_F \times \Psi_p$ with the same equivariance property. It is clear from the definition and (2.1.2) that it satisfies the required interpolation property. □

2.2. Equivalence of statements. — We now restate Theorem B in a form which directly generalises [I, Theorem B]. It is the form in which we will prove it, for convenience of reference.

We retain the setup of §1.2. Let d_F be the Γ_F-differential defined before [I, Theorem B]. For all $v \nmid \infty$ let dt_v be the measure on $E_v^\times/F_v^\times$ specified in [I, paragraph after (1.1.2)] if $v \nmid \infty$ and the measure giving $\mathbf{C}^\times/\mathbf{R}^\times$ volume 2 if $v|\infty$.

Theorem 2.2.1. — *Suppose that χ_p is not exceptional and sufficiently ramified. Fix a decomposition $(,)_\pi = \prod_v (,)_v$, with $(1,1)_v = 1$ if $v|\infty$. Then for all $f_1 \in \pi$, $f_2 \in \pi^\vee$,*

$$\langle P(f_1,\chi), P^\vee(f_2,\chi^{-1})\rangle = c_E \cdot \prod_{v|p} Z_v^\circ(\chi_v)^{-1} \cdot \mathrm{d}_F L_{p,\alpha}(\sigma_{A,E})(\chi) \cdot \prod_{v\nmid\infty} Q_{(,)_v,dt_v}(f_1,f_2,\chi) \tag{2.2.1}$$

in $\Gamma_F \hat{\otimes} L(\chi)$, where

$$c_E := \frac{\zeta_F(2)}{(\pi/2)^{[F:\mathbf{Q}]}|D_E|^{1/2} L(1,\eta)} \in \mathbf{Q}^\times.$$

Lemma 2.2.2. — *Theorem 2.2.1 is equivalent to Theorem B. When every prime $v|p$ splits in E, it specialises to* [I, Theorem B] *with the correction given in Appendix B.*

Proof. — The second assertion is immediate; we prove the first one. First, we note that (2.2.1) is equivalent to

(2.2.2)
$$\frac{\langle P(f_1,\chi), P^\vee(f_2,\chi^{-1})\rangle}{(f_3,f_4)_\pi} = c_E \cdot \prod_{v|p} Z_v^\circ(\chi_v)^{-1} \cdot \mathrm{d}_F L_{p,\alpha}(\sigma_{A,E})(\chi) \cdot 2^{-[F:\mathbf{Q}]} \prod_v \frac{Q_{(,)_v,dt_v}(f_1,f_2,\chi)}{(f_{3,v},f_{4,v})_v}$$

for any $f_3 \in \pi$, $f_4 \in \pi$ with $f_{3,\infty} = f_{4,\infty} = 1$ and $(f_3, f_4)_\pi \neq 0$ (the extra power of 2 comes from the archimedean places). The left-hand side of (2.2.2) is the same as that of the formula of Theorem B, and the product of the terms after the L-derivative in its right-hand side equals

$$2^{-[F:\mathbf{Q}]}\frac{\prod_v dt_v}{dt}\cdot Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4};\chi\right) = 2^{1-[F:\mathbf{Q}]}|D_{E/F}|^{1/2}|D_F|^{1/2}\pi^{-[F:\mathbf{Q}]}L(1,\eta)\cdot Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4};\chi\right),$$

because the measure $\prod_v dt_v$ (respectively dt) gives $E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times$ volume $2|D_{E/F}|^{1/2}|D_F|^{1/2}\pi^{-[F:\mathbf{Q}]}L(1,\eta)$ (respectively 1).

Next, we have $\mathrm{d}_F L_{p,\alpha}(\sigma_E)(\chi) = \frac{1}{2}\mathrm{d}(j_\chi^* L_{p,\alpha}(\sigma_E))(\mathbf{1})$, and it is clear from comparing the interpolation properties that

$$\prod_{v|p} Z_v^\circ(\chi_v)^{-1}\cdot\frac{1}{2}\mathrm{d}\left(j_\chi^* L_{p,\alpha}(\sigma_E)\right)(\mathbf{1}) = \frac{\pi^{2[F:\mathbf{Q}]}|D_F|^{1/2}}{2\zeta_F(2)}\cdot e_p(V_{(A,\chi)})^{-1}\cdot\mathscr{L}_p'(V_{(A,\chi)},0).$$

It follows that the right hand side of (2.2.1) equals $c\cdot e_p(V_{(A,\chi)})^{-1}\cdot\mathscr{L}_p'(V_{(A,\chi)},0)\cdot Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4};\chi\right)$, where

$$c = c_E\cdot\frac{\pi^{2[F:\mathbf{Q}]}|D_F|^{1/2}}{2\zeta_F(2)}\cdot 2^{1-[F:\mathbf{Q}]}|D_{E/F}|^{1/2}|D_F|^{1/2}\pi^{-[F:\mathbf{Q}]}L(1,\eta) = 1.$$

□

3. Structure of the proof

We review the formal structure of the proof in [I], dwelling only on those points where the arguments need to be modified or corrected. For an introductory description with some more details than given in §1.1, see [I, §1.7]. Readers interested in a detailed understanding of the present section are advised to keep a copy of [I] handy.

3.1. Notation and setup. — We very briefly review some notation and definitions from [I], which will be used throughout the paper.

Galois groups. — If K is a perfect field, we denote by $\mathscr{G}_K := \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group.

Local fields. — For v finite a place of F, we denote by ϖ_v a fixed uniformiser and by $q_{F,v}$ the cardinality of the residue field of F_v. We denote by d_v a generator of the absolute different of F_v, by D_v a generator of the relative discriminant of E_v/F_v (equal to 1 unless v ramifies in E), and by e_v the ramification degree of E_v/F_v. If $w|v$ is a place of E, we denote by $q_v\colon E_v\to F_v$ and $q_w\colon E_w\to F_v$ the relative norm maps.

Groups, measures, integration. — We adopt the same notation and choices of measures as in [I, §1.9], including a regularised integration $\int^*$. In particular $T := \mathrm{Res}_{E/F}\mathbf{G}_{m,E}$, $Z = \mathbf{G}_{m,F}$, and on the adelic points of T/Z we use two measures dt (the same as introduced above Theorem 2.2.1) and $d^\circ t$. *The measure denoted by dt in the introduction will not be used.*

Operators at p. — Let $v|p$ be a place of F. We denote by ϖ_v a fixed uniformiser at v. For $r\geq 1$ we let $K_1^1(\varpi_v^r)\subset \mathrm{GL}_2(\mathscr{O}_{F,v})$ be the subgroup of matrices which become upper unipotent upon reduction modulo ϖ^r. We denote by

$$\mathrm{U}_{v,*} = K_1^1(\varpi_v^r)\begin{pmatrix}1 & \\ & \varpi_v^{-1}\end{pmatrix}K_1^1(\varpi_v^r),\qquad \mathrm{U}_v^* = K_1^1(\varpi_v^r)\begin{pmatrix}\varpi^r & \\ & 1\end{pmatrix}K_1^1(\varpi_v^r),$$

the usual double coset operators, and by

$$w_{r,v} := \begin{pmatrix} & 1\\ -\varpi_v^r & \end{pmatrix}\in\mathrm{GL}_2(F_v).$$

We also let $w_r := \prod_{v|p} w_{r,v} \in \mathrm{GL}_2(F_p)$ and, if $(\beta_v)_{v|p}$ are characters of $F_v^\times$, we denote $\beta_p(\varpi) := \prod_{v|p} \beta_v(\varpi_v)$.

Spaces of characters. — We denote by $\mathscr{Y}_F$, $\mathscr{Y}'$, $\mathscr{Y}$ respectively the schemes over L defined in (1.2.7), (2.1.1) and the subscheme of $\mathscr{Y}$ cut out by the condition $\chi_{|\mathbf{A}^{\infty\times}} = \omega^{-1}$. We add to this notation a superscript 'l.c.' to denote the ind-subschemes of locally constant characters (which has a model over a finite extension of M in L).

Let $\mathscr{I}_{\mathscr{Y}/\mathscr{Y}'}$ be the ideal sheaf of $\mathscr{Y} \subset \mathscr{Y}'$. If $\mathscr{M}$ is a coherent $\mathscr{O}_{\mathscr{Y}'}$-module, we denote

$$\mathrm{d}_F \colon \mathscr{M} \otimes_{\mathscr{O}_{\mathscr{Y}'}} \mathscr{I}_{\mathscr{Y}/\mathscr{Y}'} \to \mathscr{M} \otimes_{\mathscr{O}_{\mathscr{Y}'}} \mathscr{I}_{\mathscr{Y}/\mathscr{Y}'}/\mathscr{I}^2_{\mathscr{Y}/\mathscr{Y}'} = \mathscr{M}_{|\mathscr{Y}} \hat{\otimes} \Gamma_F$$

the normal derivative (cf. the definition before [I, Theorems B].)

Kirillov models. — Let $\sigma^\infty = \bigotimes_{v\nmid\infty} \sigma_v$ be the M-rational automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ attached to A, and denote abusively still by σ^∞ its base-change to L. For every place v the representations σ_v of $\mathbf{B}_v^\times$ and π_v of $\mathrm{GL}_2(F_v)$ are Jacquet–Langlands correspondents.

For $v|p$, we denote by

$$\mathscr{K}_{\psi_v} \colon \sigma_v \to C^\infty(F_v^\times, L)$$

a fixed rational Kirillov model. It depends on a choice of character ψ_v, after choosing an extension of the base to a universal space Ψ_v analogous to Ψ_p. (We do not dwell on this subtlety discussed in [I, §2.3].)

Orhogonal spaces. — We let $\mathbf{V} := \mathbf{B}$ equipped with the reduced norm q, a quadratic form valued in $\mathbf{A}$. The image of $E_\mathbf{A}$ is a subspace $\mathbf{V}_1$ ot the orthogonal space $\mathbf{V}$, and we let $\mathbf{V}_2$ be its orthogonal complement. The restriction $q_{|\mathbf{V}_1}$ is the adelisation of the norm of E/F.

Schwartz spaces and Weil representation. — If $\mathbf{V}'$ is any one of the above spaces, we denote by $\overline{\mathscr{S}}(\mathbf{V}' \times \mathbf{A}^\times) = \bigotimes_v' \overline{\mathscr{S}}(\mathbf{V}_v' \times F_v^\times)$ the Fock space of Schwartz functions considered in [I]. (This differs from the usual Schwartz space only at infinity.) There is a *Weil representation*

$$r = r_\psi \colon \mathrm{GL}_2(\mathbf{A}) \times \mathrm{O}(\mathbf{V}, q) \to \mathrm{End}\, \overline{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times),$$

defined as in [I, §3.1]. The orthogonal group of $\mathbf{V}$ naturally contains the product $T(\mathbf{A}) \times T(\mathbf{A})$ acting by left and right multiplication on $\mathbf{V}$. The Weil representation also depends on a choice of additive characters ψ. The restriction $r_{|T(\mathbf{A})\times T(\mathbf{A})}$ preserves the decomposition $\mathbf{V}_1 \oplus \mathbf{V}_2$, hence it accordingly decomposes as $r_1 \oplus r_2$.

Special data at p. — We list the functions at the places $v|p$ that we use. The additive character ψ_v will be assumed to be of level d_v^{-1}.

Define

$$W_v(y) := \mathbf{1}_{\mathscr{O}_{F,v}-0}(y)|y|_v \alpha_v(y), \tag{3.1.1}$$

the ordinary vector in the fixed Kirillov model $\mathscr{K}_{\psi_v}$ of $\sigma_{A,v}$. We consider

$$\varphi_v = \varphi_{v,r} := \mathscr{K}_{\psi_v}^{-1}(\alpha_v(\varpi_v)^{-r} w_r W_v) \in \sigma_v. \tag{3.1.2}$$

Now we consider Schwartz functions. We let

$$\mathbf{B}_v \cong M_2(F_v)$$

be the indefinite quaternion algebra over F_v; this choice is justified *a posteriori* by Corollary A.2.3. The following choices of functions correct and modify the ones fixed in [I] (cf. the Errata in Appendix B); note in particular that we will use two different functions on $\mathbf{V}_{2,v}$.

Decompose orthogonally $\mathbf{V}_v = \mathbf{V}_{1,v} \oplus \mathbf{V}_{2,v}$, where $\mathbf{V}_{1,v} = E_v$ under the fixed embedding $E_{\mathbf{A}^\infty} \hookrightarrow \mathbf{B}^\infty$. We define the following Schwartz functions on, respectively, $F_v^\times$ and its product with $\mathbf{V}_{1,v}$, $\mathbf{V}_{2,v}$, $\mathbf{V}_v$:

$$\begin{aligned} &\phi_{F,r}(u) := \delta_{1,U_{F,r}}(u), && \text{where } \delta_{1,U_{F,r}}(u) := \frac{\mathrm{vol}(\mathscr{O}_{F,v}^\times)}{\mathrm{vol}(1+\varpi^r\mathscr{O}_{F,v})}\mathbf{1}_{1+\varpi^r\mathscr{O}_{F,v}}(u); \\ &\phi_{1,r}(x_1,u) := \delta_{1,U_{T,r}}(x_1)\delta_{1,U_{F,r}}(u), && \text{where } \delta_{1,U_{T,r}}(x_1) = \frac{\mathrm{vol}(\mathscr{O}_{E,v})}{\mathrm{vol}(1+\varpi^r\mathscr{O}_{E,v})}\mathbf{1}_{1+\varpi^r\mathscr{O}_{E,v}}(x_1); \end{aligned} \tag{3.1.3}$$

and

$$\begin{aligned} \phi_2^\circ(x_2,u) &:= \mathbf{1}_{\mathscr{O}_{\mathbf{V}_{2,v}}}(x_2)\mathbf{1}_{\mathscr{O}_{F,v}^\times}(u); \\ \phi_{2,r}(x_2,u) &:= e_v^{-1}|d|_v\cdot\mathbf{1}_{\mathscr{O}_{\mathbf{V}_{2,v}}\cap q^{-1}(-1+\varpi^r\mathscr{O}_{F,v})}(x_2)\mathbf{1}_{1+\varpi^r\mathscr{O}_{F,v}}(u); \\ \phi_r(x,u) &:= \phi_{1,r}(x_1,u)\phi_{2,r}(x_2,u). \end{aligned} \tag{3.1.4}$$

p-adic modular forms and q-expansions. — In [**I**, §2], we have defined the notion of Hilbert automorphic forms and twisted Hilbert automorphic forms (the latter depend on an extra variable $u\in\mathbf{A}^\times$. We have also defined the associated space of q-expansions, and a less redundant space of *reduced* q-expansions. When the coefficient field is a finite extension L' of $\mathbf{Q}_p$ these spaces are endowed with a topology. We have an (injective) reduced-q-expansion map on modular forms, denoted by

$$\varphi'\mapsto{}^{\mathbf{q}}\varphi'.$$

The image of modular forms (respectively cuspforms) of level $K^pK_1^1(p^\infty)\subset\mathrm{GL}_2(\mathbf{A}^{p\infty})$, parallel weight 2 and central character ω^{-1} is denoted by $\mathbf{M}=\mathbf{M}(K^p,\omega^{-1})$ (respectively $\mathbf{S}$). The closure of $\mathbf{M}$ (respectively $\mathbf{S}$) in the space of q-expansions with coefficients in L' is denoted $\mathbf{M}'$, respectively $\mathbf{S}'$ and its elements are called p-adic modular forms (respectively cuspforms).

If $\mathscr{Y}^?=\mathscr{Y}_F,\mathscr{Y}'$, we define the notion of a $\mathscr{Y}^?$-family of modular forms by copying word for word [**I**, Definition 2.1.3]; the resulting notion coincides with that of bounded families on the analogous rigid spaces considered in *loc. cit.*

For a finite set of places S disjoint from those above p, we have also defined a certain quotient space $\overline{\mathbf{S}}_S'$ of cuspidal reduced q-expansions modulo those all of whose coefficients of index $a\in F^\times\mathbf{A}^{\S\infty\times}$ vanish. According to [**I**, Lemma 2.1.2], for any S the reduced-q-expansion map induces an *injection*

$$\mathbf{S}\hookrightarrow\overline{\mathbf{S}}_S'. \tag{3.1.5}$$

p-adic Petersson product and p-critical forms. — For $\varphi^p\in\sigma$, we defined in [**I**, Proposition 2.4.4] a functional

$$\ell_{\varphi^p,\alpha}\colon\mathbf{M}(K^p,\omega^{-1},L)\to L,$$

whose restriction to classical modular forms equals, up to an adjoint L-value, the limit as $r\to\infty$ of Petersson products with antiholomorphic forms $\varphi^p\varphi_{p,r}\in\sigma$ with component $\varphi_{p,r}=\prod_{v|p}\varphi_{v,r}$ as in (3.1.2).

Let $v|p$. We say that a form or q-expansion over a finite-dimensional $\mathbf{Q}_p$-vector space L' is *v-critical* if its coefficients a_* (where $*\in\mathbf{A}^{\infty\times}$)

$$a_{m\varpi_v^s}=O(q_{F,v}^s) \tag{3.1.6}$$

in L', uniformly in $m\in\mathbf{A}^{\infty\times}$. Here for two functions $f,g\colon\mathbf{N}\to L$, we write

$$f=O(g)\iff\text{there is a constant } c>0 \text{ such that } |f(s)|\le c|g(s)| \text{ for all sufficiently large } s.$$

The space of *p-critical* forms is the sum of the spaces of v-critical forms for $v|p$. Any element in those spaces is annihilated by $\ell_{\varphi^p,\alpha}$.

3.2. Analytic kernel. — The analytic kernel is a p-adic family of theta-Eisenstein series, related to the p-adic L-function. We review its main properties.

Proposition 3.2.1. — *There exist p-adic families of q-expansions of modular forms $\mathscr{E}$ over $\mathscr{Y}_F$ and $\mathscr{I}$ over $\mathscr{Y}'$, satisfying:*

1. *For any $\chi_F \in \mathscr{Y}_F^{\rm l.c.}(\mathbf{C})$ and any $r = (r_v)_{v|p}$ satisfying $c(\chi_F)|p^r$, we have the identity of q-expansions of twisted modular forms of weight 1:*

$$\mathscr{E}(u, \phi_2^{p\infty}; \chi_F) = |D_F| \frac{L^{(p)}(1, \eta\chi_F)}{L^{(p)}(1, \eta)}\, {}^{\mathbf{q}}E_r(u, \phi_2, \chi_F),$$

where

$$E_r(g, u, \phi_2, \chi_F) := \sum_{\gamma \in P^1(F)\backslash SL_2(F)} \delta_{\chi_F, r}(\gamma g w_r) r(\gamma g)\phi_2(0, u) \tag{3.2.1}$$

is the Eisenstein series defined in [I, §3.2], *with respect to $\phi_2 = \phi_2^{p\infty}(\chi_F)\phi^\circ_{2,p\infty}$ with $\phi^\circ_{2,v}$ as in* (3.1.4) *for $v|p$, and $\phi_{2,v}(\chi_v)$ for $v \nmid \infty$ and $\phi^\circ_{2,v}$ for $v|\infty$ as defined in* loc. cit.

2. *For $\phi_1 \in \overline{\mathscr{S}}(\mathbf{V}_1 \times \mathbf{A}^\times)$ and $\chi' \in \mathscr{Y}'^{\rm l.c.}$, consider the twisted modular form of weight 1 with parameter $t \in E_{\mathbf{A}}^\times$:*

$$\theta(g, (t, 1)u, \phi_1) := \sum_{x_1 \in E} r_1(g, (t, 1))\phi_1(x, u). \tag{3.2.2}$$

For any $\chi' \in \mathscr{Y}'^{\rm l.c.}$, let $\chi_F := \omega^{-1}\chi'_{|\mathbf{A}^\times} \in \mathscr{Y}_F^{\rm l.c.}$. Then for any $r = (r_v)_{v|p}$ satisfying $r_v \geq 1$ and $c(\chi_F)|p^r$ we have

$$\mathscr{I}(\phi^{p\infty}; \chi') = \frac{c_{U^p}|D_E|^{1/2}}{|D_F|^{1/2}} \int^*_{[T]} \chi'(t) \sum_{u \in \mu^2_{U^p}\backslash F^\times} {}^{\mathbf{q}}\theta((t, 1), u, \phi_1; \chi')\mathscr{E}(q(t)u, \phi_2^{p\infty}; \chi_F)\, d^\circ t, \tag{3.2.3}$$

where for $v|p$, $\phi_{1,v} = \phi_{1,v,r}$ is as in (3.1.3).

3. *We have*

$$\ell_{\varphi^p, \alpha}(\mathscr{I}(\phi^{p\infty})) = L_{p,\alpha}(\sigma_E) \cdot \prod_{v|p} |d|_v^2 |D|_v \prod_{v \nmid p\infty} \mathscr{R}_v^\natural(W_v, \phi_v, \chi'_v) \tag{3.2.4}$$

where the local terms $\mathscr{R}_v^\natural$ are as in [I, Propositions 3.5.1, 3.6.1].

Proof. — Part 1 is [I, Proposition 3.3.2]. Part 2 summarises [I, §3.4]. Part 3 is [I, (3.7.1)] with the correction of Appendix B below. □

Derivative of the analytic kernel. — We denote

$$\mathscr{I}'(\phi^{p\infty}; \chi) := {\rm d}_F \mathscr{I}(\phi^{p\infty}; \chi), \tag{3.2.5}$$

a p-adic modular form with coefficients in $\Gamma_F \hat{\otimes} L(\chi)$.

3.3. Geometric kernel. — The geometric kernel function, see [I, §§5.2-5.3], is related to heights of Heegner points. We recall its construction and modularity.

For any $x \in \mathbf{B}^{\infty\times}$, we have a Hecke correspondence $Z(x)_U$ on $X_U \times X_U$. For $a \in \mathbf{A}^{\infty\times}$, $\phi^\infty \in \overline{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$, consider the correspondences

$$\widetilde{Z}_a(\phi^\infty) := c_{U^p} w_U |a| \sum_{x \in U\backslash \mathbf{B}^{\infty\times}/U} \phi^\infty(x, aq(x)^{-1})Z(x)_U \tag{3.3.1}$$

where $w_U = |\{\pm 1\} \cap U|$ and c_{U^p} is defined in [I, (3.4.3)]. By [I, Theorem 5.2.1] (due to Yuan–Zhang–Zhang), their images in $\mathrm{Pic}(X_U \times X_U)_{\mathbf{Q}}$ are the reduced q-expansion coefficients different from the constant term of an automorphic form.

Fix any $P \in X^{E^\times}(E^{\rm ab})$, and for $t \in E^\times_{\mathbf{A}^\infty}$ let $[t]$ be the Hecke-translate of P by t. Let $\mathrm{Cl}(X_{U,\overline{F}})_{\mathbf{Q}} \supset \mathrm{Cl}^0(X_{U,\overline{F}})_{\mathbf{Q}}$ be the space of divisor classes with $\mathbf{Q}$-coefficients and, respectively, its subspace consisting of classes with degree 0 on every connected component. Denote by $(\ )^0 \colon \mathrm{Cl}(X_{U,\overline{F}})_{\mathbf{Q}} \to \mathrm{Cl}^0(X_{U,\overline{F}})_{\mathbf{Q}}$ the linear section of the inclusion whose kernel is spanned by the pushforwards to $X_{U,\overline{F}}$ of the classes of the canonical bundles of the connected components of $X_{U',\overline{F}}$, for any sufficiently small U'.

Let

$$\theta_{\iota_{\mathfrak{p}}} \colon (\sigma^\infty \otimes \overline{\mathscr{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty,\times})) \otimes_M L \to (\pi \otimes \pi^\vee) \otimes_{M,\iota_{\mathfrak{p}}} L$$

be Shimizu's theta lifting defined in [I, §5.1]. Let

$$\mathrm{T}_{\mathrm{alg}} \colon \pi^U \otimes_M \pi^{\vee,U} \to \mathrm{Hom}\,(J_U, J_U^\vee) \otimes M$$

be defined by $\mathrm{T}_{\mathrm{alg}}(f_1,f_2) := f_2^\vee \circ f_1$. Finally, let

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_X \colon J^\vee(\overline{F}) \times J(\overline{F}) \to \Gamma_F \hat{\otimes} L \tag{3.3.2}$$

be the p-adic height pairing defined as in[(5)] [I, Lemma 5.3.1]. (We abusively omit the subscript X as we will no longer need to use the pairing on $A_E(\chi) \otimes A_E^\vee(\chi)$.)

Proposition 3.3.1. — *The series*

$$\widetilde{Z}(\phi^\infty, \chi) := \int^*_{[T]} \chi(t) \left(\sum_{a \in F^\times} \langle \widetilde{Z}_a(\phi^\infty)[1]^0, [t^{-1}]^0 \rangle\, \mathbf{q}^a \right) d^\circ t$$

does not depend on the choice of U, and it is (the q-expansion of) a weight-2 cuspidal Hilbert modular form of central character ω^{-1}, with coefficients in $\Gamma_F \hat{\otimes} L(\chi)$.

If $\phi_v = \phi_{r,v}$ is an in (3.1.4) for all $v|p$, then for any sufficiently large r', the form $\widetilde{Z}(\phi^\infty,\chi)$ is invariant under $\prod_{v|p} K_1^1(\varpi_v^{r'})$ and it satisfies

$$\ell_{\varphi^p,\alpha}(\widetilde{Z}(\phi^\infty,\chi)) = 2|D_F|^{1/2}|D_E|^{1/2}L(1,\eta)\langle \mathrm{T}_{\mathrm{alg},\iota_{\mathfrak{p}}}(\theta_{\iota_{\mathfrak{p}}}(\varphi, \alpha_p|\cdot|_p(\varpi)^{-r'}w_{r'}^{-1}\phi))P_\chi, P_\chi^{-1}\rangle_X. \tag{3.3.3}$$

Proof. — We have already recalled the modularity. The invariance under $\prod_{v|p} K_1^1(\varpi_v^{r'})$ follows from the invariance of ϕ_r under the action of $\left(\begin{smallmatrix} 1 & \mathscr{O}_{F,p} \\ & 1 \end{smallmatrix}\right)$ and the continuity of the Weil representation. Finally, the proof of (3.3.3) is indicated in [I, proof of Proposition 5.4.3] (with the correction of Appendix B). □

3.4. Kernel identity. — We state our kernel identity and recall how it implies the main theorem.

Assumptions on the data. — Consider the following local assumptions on the data at primes above p.

Assumption 3.4.1. — Let $U_{F,v}^\circ := 1 + \varpi_v^n \mathscr{O}_{F,v}$ with $n \geq 1$ be such that ω_v is invariant under $U_{F,v}^\circ$. The character χ_p is *sufficiently ramified* in the sense that it is nontrivial on

$$V_p^\circ := \prod_{v|p} q^{-1}_{|\mathscr{O}_{E,v}}(U_{F,v}^\circ) \subset \mathscr{O}_{E,p}^\times.$$

Assumption 3.4.2. — For each $v|p$, the open compact $U_v \subset \mathbf{B}_v^\times$ satisfies:

- $U_v = U_{v,r} = 1 + \varpi^r M_2(\mathscr{O}_{F_v,r'})$ for some $r \geq 1$;
- the integer $r \geq n$ is sufficiently large so that the characters χ_v and $\alpha_v \circ q_v$ of $E_v^\times$ are invariant under $U_{v,r} \cap \mathscr{O}_{E,v}^\times$.

(5) There is a typo in *loc. cit.* (also noted in Appendix B below): the left-hand side of the last equation in the statement should be $\langle f_1'(P_1), f_2'(P_2)\rangle_{J,*}$.

Convention on citations from [I]. — In [I], we have denoted by $S_{\text{non-split}}$ the set of places of F non-split in E, and by S_p the set of places of F above p. When referring to results from [I], we henceforth convene that one should read any assumption such as let $v \in S_{\text{non-split}}$' or 'let v be a place in F non-split in E' as 'let $v \in S_{\text{non-split}} - S_p$'. Similarly, the set S_1 fixed in [I, §6.1] should be understood to consist only of places not above p.

***Theorem 3.4.3* (Kernel identity).** — *Suppose that U, φ^p, $\phi^{p\infty}$, χ, r satisfy the assumptions of* [I, §6.1] *as well as Assumptions 3.4.1, 3.4.2. Let $\phi_p := \otimes_{v|p}\phi_{v,r}$ with $\phi_{v,r} =$(3.1.4). Then*

$$\ell_{\varphi^p,\alpha}(\mathrm{d}_F \mathscr{I}(\phi^{p\infty};\chi)) = 2|D_F|L_{(p)}(1,\eta)\cdot \ell_{\varphi^p,\alpha}(\widetilde{Z}(\phi^{\infty},\chi)).$$

The elements of the proof will be gathered in §3.6.

Corollary 3.4.4. — *Suppose that χ satisfies Assumption 3.4.1. Then Theorem 2.2.1 holds.*

Proof. — As in [I, Proposition 5.4.3] corrected in Appendix B, we consider the following equivalent (by [I, Lemma 5.3.1]) form of the identity of Theorem 2.2.1:

(3.4.1)

$$\langle \mathrm{T}_{\mathrm{alg},\iota_{\mathfrak{p}}}(f_1\otimes f_2)P_\chi, P_{\chi^{-1}}\rangle_J = \frac{\zeta_F^{\infty}(2)}{(\pi^2/2)^{[F:\mathbf{Q}]}|D_E|^{1/2}L(1,\eta)}\prod_{v|p} Z_v^{\circ}(\alpha_v,\chi_v)^{-1}\cdot \mathrm{d}_F L_{p,\alpha}(\sigma_{A,E})(\chi)\cdot Q(f_1,f_2,\chi)$$

where $\iota_{\mathfrak{p}}\colon M\hookrightarrow L(\chi)$, and $P_\chi = \fint_{[T]}\mathrm{T}_t(P-\xi_P)\chi(t)\,dt \in J(\overline{F})_{L(\chi)}$. By linearity, (3.4.1) extends to an identity which makes sense for any element $\mathbf{f}\in\pi\otimes\pi^\vee$. By the multiplicity-one result for $E^\times_{\mathbf{A}^\infty}$-invariant linear functionals on each of π, $\pi^\vee$, it suffices to prove (3.4.1) for one element $\mathbf{f}\in\pi\otimes\pi^\vee$ such that $Q(\mathbf{f},\chi)\neq 0$ (cf. [YZZ12, Lemma 3.23]).

We *claim* that Theorem 3.4.3 gives (3.4.1) for $\mathbf{f} = \theta(\varphi,\phi)$, where:

- φ_∞ is standard antiholomoprhic in the sense of [I], ϕ_∞ is standard in the sense of [I];
- for all $v|p$, $\varphi_v =$(3.1.2) and $\phi_v = \phi_{v,r} =$(3.1.4) for any sufficiently large r.

The claim follows from (3.2.4), (3.3.3), the local comparison between $\mathscr{R}_v^\natural$ and $Q_v\circ\theta_v$ for $v\nmid p$ of [I, Lemma 5.1.1], and the local calculation at $v|p$ of Proposition A.2.2.

Finally, the existence of φ,ϕ satisfying both the required assumptions and $Q(\theta(\varphi,\phi))\neq 0$ follows from [I, Lemma 6.1.6] away from p, and the explicit formula of Proposition A.2.2 at p. □

3.5. Derivative of the analytic kernel. — We start by studying the incoherent Eisenstein series $\mathscr{E}(\phi_2^{p\infty})$. For $a\in F_v^\times$, denote by

$$W^{\circ}_{a,v}$$

the normalised local Whittaker function of $E_r(\phi_2^{p\infty}(\chi_F)\phi^{\circ}_{2,p},\chi_F) =$ (3.2.1), defined as in [I, Proposition 3.2.1 and paragraph following its proof].

The following reviews and corrects [I, Proposition 7.1.1].

Proposition 3.5.1. — *For each $v|p$, let $\phi_{2,v} = \phi_{2,r,v}$ be as in* (3.1.4).

1. *Let v be a place of F and $a\in F_v^\times$.*
   (a) *If a is not represented by $(\mathbf{V}_{2,v},uq)$ then $W^{\circ}_{a,v}(g,u,\mathbf{1}) = 0$.*
   (b) *(Local Siegel–Weil formula.) If $v\nmid p$ and there exists $x_a\in\mathbf{V}_{2,v}$ such that $uq(x_a) = a$, then*

$$W^{\circ}_{a,v}(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right),u,\mathbf{1}) = \int_{E_v^1} r(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right),h)\phi_{2,v}(x_a,u)\,dh$$

(c) (Local Siegel–Weil formula at p.) *If* $v|p$, $a \in -1+\varpi^r\mathscr{O}_{F,v}$, $u \in 1+\varpi^r\mathscr{O}_{F,v}$, *let* $x_a \in \mathbf{V}_{2,v}$ *be such that* $uq(x_a)=a$. *Then*

$$W^{\circ}_{a,v}\left(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right), u, \mathbf{1}\right) = |d|_v \int_{E_v^1} r\left(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right), h\right)\phi_{2,v}(x_a, u)\, dh. \tag{3.5.1}$$

2. For any $a \in F_v^{\times} \cap \prod_{v|p}(-1+\varpi^r\mathscr{O}_{F,v})$, $u \in F^{\times}\cap\prod_{v|p}(1+\varpi^r\mathscr{O}_{F_v})$, *there is a place* $v\nmid p$ *of* F *such that* a *is not represented by* $(\mathbf{V}_2, uq)$.

Proof. — Parts 1(a)-(b) are as in [I, Proposition 7.1.1]. Before continuing, observe that, under our assumptions, a is always represented by $(\mathbf{V}_{2,v}, uq_v)$ for all $v|p$: this is clear if v splits in E and, up to possibly enlarging the integer r, it may be seen by the local constancy of q_v and the explicit identity $q_v(\mathrm{j}_v)=-1$, where j_v=(4.1.2) below if v is non-split. Then part 1(c) follows by explicit computation of both sides (starting e.g. as in [YZZ12, proof of Proposition 6.8] for the left-hand side). Explicitly, we have

$$(3.5.1) = \begin{cases} e_v^{-1}|d|_v \mathrm{vol}(E_v^1\cap\mathscr{O}_{E,v}^{\times}, dh)|y|^{1/2}\mathbf{1}_{\mathscr{O}_{F,v}}(ay)\mathbf{1}_{\mathscr{O}_{F,v}^{\times}}(y^{-1}u) & \text{if } v(a)\geq 0 \text{ and } v(u)=0 \\ 0 & \text{otherwise.}\end{cases}$$

Finally, part 2 follows from the observation of the previous paragraph and [YZZ12, Lemma 6.3]. □

Lemma 3.5.2. — *Suppose that, for all* $v|p$, $\phi_{1,v}=\phi_{1,r,v}$ *is as in* (3.1.3). *Then, for any* $t\in T(\mathbf{A})$, *the* a^{th} *q-expansion coefficient of the the theta series of* (3.2.2) *vanishes unless*

$$a \in \bigcap_{v|p} 1+\varpi^r\mathscr{O}_{F,v}.$$

Proof. — Straightforward. □

Denote

$$\mathrm{U}_{p,*}^r := (\prod_{v|p}\mathrm{U}_{v,*})^r,$$

an operator on modular forms which extends to an operator on all the spaces of p-adic q-expansions defined so far.

Corollary 3.5.3. — *Suppose that* $\phi^{p\infty}\in\mathscr{S}(\mathbf{V}_2^{p\infty}\times\mathbf{A}^{p\infty,\times})$ *satisfies the assumptions of* [I, §6.1]. *Let* $\chi\in\mathscr{Y}_{\omega}^{\text{l.c.}}$. *For any sufficiently large* r, *we have*

$$\mathrm{U}_{p,*}^r\mathscr{I}(\phi^{p\infty};\chi)=0.$$

Proof. — For the first assertion, we need to show that the a^{th} reduced q-expansion coefficient of $\mathscr{I}$ vanishes for all a satisfying $v(a)\geq r$ for all $v|p$. By the defining property 3.2.3 of $\mathscr{I}(\chi')$ and the choice of ϕ_p, the group $T(F_p)\subset T(\mathbf{A})$ acts trivially on the q-expansion coefficients of $\mathscr{I}(\chi')$. The remaining integration on $T(F)Z(\mathbf{A})V^p\backslash T(\mathbf{A})/T(F_p)$ is a finite sum, so the coefficients a is a sum of products of the coefficients of index a_1 of θ and of index a_2 of $\mathscr{E}(\mathbf{1})$, for pairs (a_1,a_2) with $a_1+a_2=a$. When $a=0$, the vanishing follows from the vanishing of the constant term of $\mathscr{E}$, which is proved as in [YZZ12, Proposition 6.7]. For $a\neq 0$, by Lemma 3.5.2 only the pairs (a_1,a_2) with $a_1\in\bigcap_{v|p}1+\varpi^r\mathscr{O}_{F,v}$ contribute. If $v(a)\geq r$ for $v|p$, this forces $a_2\in\bigcap_{v|p}-1+\varpi_v^r\mathscr{O}_{F,v}$. Then the coefficient of index a_2 of $\mathscr{E}(\mathbf{1})$ vanishes by Proposition 3.5.1. □

We can now proceed as in [I, §7.2], except for the insertion of the operator $\mathrm{U}_{p,*}^r$. (This will be innocuous for the purposes of Theorem 3.4.3, since the kernel of $\mathrm{U}_{p,*}^r$ is contained in the kernel of

$\ell_{\varphi^p,\alpha}$.) We obtain, under the assumptions of [I, §6.1], a decomposition of (3.2.5)

$$\mathrm{U}^r_{p,*}\mathscr{I}'(\phi^{p\infty};\chi)=\sum_{v\in S_{\text{non-split}}-S_p}\mathrm{U}^r_{p,*}\mathscr{I}'(\phi^{p\infty};\chi)(v)$$

valid in the space $\mathbf{S}$ of p-adic q-expansions with coefficients in $\Gamma_F\hat{\otimes}L(\chi)$. See *loc. cit.* for the definition of $\mathscr{I}'(\phi^{p\infty};\chi)(v)$.

3.6. Decomposition of the geometric kernel and comparison. — Suppose that Assumptions 3.4.1, 3.4.2 as well as the assumptions of [I, §6.1] are satisfied. Then we may decompose (see [I, §8.2]) the generating series of Proposition 3.3.1

$$\widetilde{Z}(\phi^\infty,\chi)=\sum_v \widetilde{Z}(\phi^\infty,\chi)(v),$$

according to the decomposition $\langle\ ,\ \rangle_X=\sum_v\langle\ ,\ \rangle_{X,v}$ of the height pairing. Here the sum runs over all finite places of F.

The following is the main result of [I] on the local comparison away from p. Let $\overline{\mathbf{S}}'=\overline{\mathbf{S}}'_{S_1}$ be the quotient of the space of p-adic q-expansions recalled above (3.1.5).

Theorem 3.6.1 **([I, Theorem 8.3.2]).** — *Let* $\phi^\infty=\phi^{p\infty}\phi_p$ *with* $\phi_p=\prod_{v|p}\phi_v$ *as in* (3.1.4). *Suppose that Assumptions 3.4.1, 3.4.2 as well as the assumptions of* [I, §6.1] *are satisfied. Then we have the following identities of reduced q-expansions in* $\overline{\mathbf{S}}'$*:*

1. *If* $v\in S_{\text{split}}-S_p$*, then*
$$\widetilde{Z}(\phi^\infty,\chi)(v)=0.$$
2. *If* $v\in S_{\text{non-split}}-S_1-S_p$*, then*
$$\mathrm{U}^r_{p,*}\mathscr{I}'(\phi^{p\infty};\chi)(v)=2|D_F|L_{(p)}(1,\eta)\mathrm{U}^r_{p,*}\widetilde{Z}(\phi^\infty,\chi)(v).$$
3. *If* $v\in S_1$*, then*
$$\mathrm{U}^r_{p,*}\mathscr{I}'(\phi^{p\infty};\chi)(v),\qquad \mathrm{U}^r_{p,*}\widetilde{Z}(\phi^\infty,\chi)(v)$$
are theta series attached to a quaternion algebra over F.
4. *The sum*
$$\mathrm{U}^r_{p,*}\widetilde{Z}(\phi^\infty,\chi)(p):=\sum_{v\in S_p}\mathrm{U}^r_{p,*}\widetilde{Z}(\phi^\infty,\chi)(v)$$
belongs to the isomorphic image $\overline{\mathbf{S}}\subset\overline{\mathbf{S}}'$ *of the space of p-adic modular forms* $\mathbf{S}$.

By this theorem and the decompositions of $\mathscr{I}'$ and $\widetilde{Z}$, the proof of the kernel identity of Theorem 3.4.3 (hence of the main theorem) is now reduced to showing the following proposition. (See [I, §8.3, last paragraph] for the details of the deduction.)

Proposition 3.6.2. — *Under the assumptions of Theorem 3.6.1, the p-adic modular form*

$$\mathrm{U}^r_{p,*}\widetilde{Z}(\phi^\infty,\chi)(p)\in\overline{\mathbf{S}}$$

is annihilated by $\ell_{\varphi^p,\alpha}$.

Let S be a finite set of non-archimedean places of F such that, for all $v\notin S$, all the data are unramified, U_v is maximal, and ϕ_v is standard. Let $K=K^pK_p$ be the level of the modular form $\widetilde{Z}(\phi^\infty)$, and let

$$T_{\iota_{\mathfrak{p}}}(\sigma^\vee)\in\mathscr{H}^S(L)=\mathscr{H}^S(M)\otimes_{M,\iota_{\mathfrak{p}}}L$$

be any $\sigma^\vee$-idempotent in the Hecke algebra as in [I, Proposition 2.4.4] By that result, in order to establish Proposition 3.6.2 suffices to prove that

$$\ell_{\varphi^p,\alpha}(\mathrm{U}^r_{p,*}T_{\iota_\mathfrak{p}}(\sigma^\vee)\widetilde{Z}(\phi^\infty,\chi)(p))=0. \tag{3.6.1}$$

As in [I], we will in fact prove the following, which implies (3.6.1).

Proposition 3.6.3. — *Let $v|p$. Under the assumptions of Theorem 3.6.1, for all $v|p$, the element*

$$T_{\iota_\mathfrak{p}}(\sigma^\vee)\widetilde{Z}(\phi^\infty,\chi)(v)\in\overline{\mathbf{S}}'$$

is v-critical in the sense of (3.1.6).

Its proof will occupy the following section.

4. Local heights at p

The goal of this section is to prove Proposition 3.6.3, whose assumptions we retain throughout except for an innocuous modification to the data at the places $v|p$. Namely, let $\widetilde{\phi}_v$ be the Schwartz function denoted by ϕ_r in (3.1.4), and let $\widetilde{U}_v\subset\mathbf{B}_v^\times$ be the open compact subgroup denoted by $U_{v,r}$ in Assumption 3.4.2. Then we let

$$U_v:=\widetilde{U}_vU^\circ_{F,v},\qquad \phi_v:=\fint_{U^\circ_{F,v}}r(z,1)\widetilde{\phi}_v\,dz.$$

Since $\chi_{v|F_v^\times}=\omega_v^{-1}$ is by construction invariant under $U^\circ_{F,v}$, the geometric kernels $\widetilde{Z}(\phi^{p\infty}\widetilde{\phi}_p,\chi)$ and $\widetilde{Z}(\phi^{p\infty}\phi_p,\chi)$ are equal. Therefore we may work on the curve X_U.

We fix a place $v|p$. If v splits in E then the desired result is proven in [I, §9]. Therefore we may and do assume that v is non-split.

4.1. Norm relation for the generating series. — The goal of this subsection will be to show that, for s large enough, each q-expansion coefficient of $\mathrm{U}^s_{p,*}\widetilde{Z}(\phi^\infty,\chi)$ is a height pairing of CM divisors of which one is supported on Galois orbits of CM points of 'pseudo-conductor' s (as defined below).

We start by considering the $\mathrm{U}_{v,*}$-action on the generating series $\widetilde{Z}(\phi^\infty)$.

Lemma 4.1.1. — *If $v(a)\geq -v(d)$, the a^{th} reduced q-expansion coefficient of* $\mathrm{U}_{v,*}\widetilde{Z}(\phi^\infty)$ *equals*

$$\widetilde{Z}_{a\varpi_v}(\phi^\infty),$$

where $\widetilde{Z}_{a'}(\phi^\infty)$ *is defined in* (3.3.1).

Proof. — After computing the Weil action of $\mathrm{U}_{v,*}$ on ϕ, this is a simple change of variables. □

We have

$$\widetilde{Z}_a(\phi^\infty)=\widetilde{Z}_{a^v}(\phi^{v\infty})Z_{a_v}(\phi_v)$$

where

$$\widetilde{Z}^v_a(\phi^v):=c_{U^p}\sum_{x^v\in U^v\backslash\mathbf{B}^{v\infty\times}/U^v}\phi^{v\infty}(x^v,aq(x^v)^{-1})Z(x^v)_U,$$

$$Z_{a_v}(\phi_v):=\sum_{x_v\in U_v\backslash\mathbf{B}_v^\times/U_v}\phi_v(x_v,aq(x_v)^{-1})Z(x_v)_U. \tag{4.1.1}$$

From here until after the proof of Lemma 4.1.3, we work in a local situation and drop v from the notation. Let $\theta\in\mathscr{O}_E$ be such that $\mathscr{O}_E=\mathscr{O}_F+\theta\mathscr{O}_F$, and write $\mathrm{T}=\mathrm{Tr}_{E/F}(\theta)$, $\mathrm{N}=\mathrm{N}_{E/F}(\theta)$.

Fix the embedding $E \to \mathbf{B} = M_2(F)$ to be

$$t = a + \theta b \mapsto \begin{pmatrix} a + b\mathrm{T} & b\mathrm{N} \\ -b & a \end{pmatrix}.$$

Let

$$\mathrm{j} := \begin{pmatrix} 1 & \mathrm{T} \\ & -1 \end{pmatrix}. \tag{4.1.2}$$

Then $\mathrm{j}^2 = 1$, $q(\mathrm{j}) = -1$, and for all $t \in E$, $\mathrm{j}t = t^c\mathrm{j}$; and in the orthogonal decomposition $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$, we have

$$\mathscr{O}_{\mathbf{V}_2} = \mathbf{V}_2 \cap M_2(\mathscr{O}_F) = \mathrm{j}\mathscr{O}_E.$$

Let $\Xi(\varpi^r) = 1 + \varpi^r\mathscr{O}_E + \mathrm{j}(\mathscr{O}_E \cap q^{-1}(1 + \varpi^r\mathscr{O}_F)$; then ϕ is a fixed multiple of the characteristic function of $\Xi(\varpi^r) \times (1 + \varpi^r\mathscr{O}_F) \subset \mathbf{B} \times F^\times$. For $a \in F^\times$, let

$$\Xi(\varpi^r)_a = \{x \in \Xi(\varpi^r) \mid q(x) \in a(1 + \varpi^r\mathscr{O}_F)\}.$$

Then the local component (4.1.1) of the generating series equals

$$Z_a(\phi) = \sum_{x \in U\backslash \Xi(\varpi^r)_a U_F^\circ/U} Z(x)_U,$$

up to a constant which is independent of a.

Lemma 4.1.2. — *Let $a \in F^\times$ satisfy $v(a) = s \geq r$. The natural map $\Xi(\varpi^r)_a U_F^\circ/U \to U\backslash\Xi(\varpi^r)_a U_F^\circ/U$ is a bijection. For either quotient set, a complete set of representatives is given by the elements*

$$x(b) := 1 + \mathrm{j}b$$

as b ranges through a complete set of representatives for

$$q^{-1}(1 - a(1 + \varpi^r\mathscr{O}_F))/(1 + \varpi^{r+s}\mathscr{O}_E) \subset (\mathscr{O}_E/\varpi^{r+s}\mathscr{O}_E)^\times.$$

Proof. — It is equivalent to prove the same statement for the quotients of $\Xi(\varpi^r)_a$ by the group $\widetilde{U} = 1 + \varpi^r M_2(\mathscr{O}_F)$. By acting on the right with elements of $\mathscr{O}_E \cap U$, we can bring any element of Ξ to one of the form $x(b)$. Write any $\gamma \in \widetilde{U}$ as $\gamma = 1 + \varpi^r u_1 + \mathrm{j}\varpi^r u_2$ with u_1, $u_2 \in \mathscr{O}_E$. Then

$$x(b)\gamma = (1 + \mathrm{j}b)(1 + \varpi^r u_1 + \mathrm{j}\varpi^r u_2) = 1 + \varpi^r(u_1 + b^c u_2) + \mathrm{j}(b + \varpi^r(bu_1 + u_2)$$

is another element of the form $x(b')$ if and only if $u_1 = -b^c u_2$. In this case

$$b' = b + \varpi^r(1 - q(b))u_2 \in b(1 + \varpi^{r+s}\mathscr{O}_E).$$

Thus the class of b modulo ϖ^{r+s} is the only invariant of the quotient $\Xi/\widetilde{U}$. The $\widetilde{U}$-action on the left similarly preserves this invariant. □

Denote by rec_E the reciprocity map of class field theory.

Lemma 4.1.3. — *Fix $a \in \mathscr{O}_F$ with $v(a) = s \geq r$.*

1. *Let $b \in (\mathscr{O}_E/\varpi^{r+s}\mathscr{O}_E)^\times$, $t \in 1 + \varpi^r\mathscr{O}_E$. Then*

$$\mathrm{rec}_E(t)[x(b)]_U = [x(bt^c/t)]_U.$$

2. *We have*

$$\Xi(\varpi^r)_a U_F^\circ/U = \bigsqcup_{\overline{b}} \mathrm{rec}_E((1 + \varpi^r\mathscr{O}_E)/(1 + \varpi^r\mathscr{O}_F)(1 + \varpi^{r+s}\mathscr{O}_E))[x(\overline{b})]_U,$$

where the Galois action is faithful, and $\overline{b}$ ranges through a set of representatives for

$$q^{-1}(1-a(1+\varpi^r\mathscr{O}_F))/(1+\varpi^{r+s}\mathscr{O}_E)\cdot(1+\varpi^{r+v(\theta-\theta^c)}\mathscr{O}_E\cap q^{-1}(1)). \tag{4.1.3}$$

The size of the set (4.1.3) *is bounded uniformly in a.*

Proof. — For part 1, we have

$$\mathrm{rec}_E(t)[x(b)]_U=[t+t\mathrm{j}b]_U=[t+\mathrm{j}t^cb]_U=[1+\mathrm{j}bt^c/t]_U.$$

Part 2 follows Lemma 4.1.2 and part 1, noting that the group $(1+\varpi^{r+v(\theta-\theta^c)}\mathscr{O}_E\cap q^{-1}(1))$ is the image of the map $t\mapsto t^c/t$ on $1+\varpi^r\mathscr{O}_E$. Finally, the map (projection, q) gives an injection from

$$q^{-1}(1-a(1+\varpi^r\mathscr{O}_F))/(1+\varpi^{r+s}\mathscr{O}_E)\cdot(1+\varpi^{r+v(\theta-\theta^c)}\mathscr{O}_E\cap q^{-1}(1))$$

to

$$(\mathscr{O}_E/\varpi^{r+v(\theta-\theta^c)}\mathscr{O}_E)^\times\times(1-a(1+\varpi^r\mathscr{O}_F))/q(1+\varpi^{r+s}\mathscr{O}_E),$$

whose size is bounded uniformly in a (more precisely, the second factor is isomorphic to $\mathscr{O}_F/\mathrm{Tr}(\mathscr{O}_E)$ via the map $1-a(1+\varpi^rx)\mapsto x$). □

We go back to a global setting and notation, restoring the subscripts v, and we denote by $\overline{w}$ an extension of the place v to E^{ab}. For $s\geq 0$, let

$$H_s\subset E^{\mathrm{ab}}$$

be the finite abelian extension of E with norm group $U_F^\circ U_T^v(1+\varpi^{r_v+s}\mathscr{O}_E)$, where $U_T^v=U^v\cap E^\times_{\mathbf{A}^\infty}$. Let $H_\infty=\bigcup_{s\geq0}H_s$. If r_v is sufficiently large, for all $s\geq 0$ the extension H_s/H_0 is totally ramified at $\overline{w}$ of degree $q_{F,v}^s$, and

$$\mathrm{Gal}(H_s/H_0)\cong\mathrm{Gal}(H_{s,\overline{w}}/H_{0,\overline{w}})\cong(1+\varpi_v^{r_v}\mathscr{O}_{E,v})/(1+\varpi_v^{r_v}\mathscr{O}_{F,v})(1+\varpi^{r_v+s}\mathscr{O}_{E,v}).$$

In particular,

$$[H_s:H_0]=[H_{s,\overline{w}}:H_{0,\overline{w}}]=q_{F,v}^s.$$

We will say that a CM point $z\in X_{H_{0,\overline{w}}}$ has *pseudo-conductor* $s\geq 0$ if $H_{0,\overline{w}}(z)=H_{s,\overline{w}}$.

Proposition 4.1.4. — *There exists an integer $d>0$ such that for all $a\in\mathbf{A}^{\infty,x}$ with $v(a)=s\geq r_v$, there exists a degree-zero divisor $D_a\in d^{-1}\mathrm{Div}^0(X_{U,H_s})$, supported on CM points, such that*

$$\widetilde{Z}_a(\phi^\infty)[1]_U=\mathrm{Tr}_{H_s/H_0}(D_a)$$

in $\mathrm{Div}^0(X_{U,H_0})$. *All prime divisor components of D_a are CM points of pseudo-conductor s.*

Proof. — This follows from Lemma 4.1.3.2, by taking D_a to be a fixed rational multiple (independent of a) of

$$\widetilde{Z}_a^v(\phi^{v\infty})Z(\omega_v)\sum_{\overline{b}\in(4.1.3)}[x(\overline{b})]_U.$$

The divisor is of degree zero by [I, Proposition 8.1.1]. Its prime components are not defined over proper subfields $H_{s'}\subset H_s$ because of the faithfulness statement of Lemma 4.1.3.2. □

4.2. Intersection multiplicities on arithmetic surfaces. — Before continuing, we gather some definitions and a key result.

Ultrametricity of intersections on surfaces. — Let $\mathscr{X}$ be a 2-dimensional regular Noetherian scheme, finite flat over a field κ or a discrete valuation ring $\mathscr{O}$ with residue field κ. We denote by $(\,\cdot\,)_{\mathscr{X}}$ the usual bilinear intersection multiplicity of divisors intersecting properly on $\mathscr{X}$, defined for effective divisors D_j $(j=1,2)$ with $\mathscr{O}_{D_j}=\mathscr{O}_{\mathscr{X}}/\mathscr{I}_j$

$$(D_1\cdot D_2)_{\mathscr{X}}=\mathrm{length}_\kappa \mathscr{O}_{\mathscr{X}}/\mathscr{I}_1+\mathscr{I}_2.$$

The subscript $\mathscr{X}$ will be omitted when clear from context.

We will need the following result of García Barroso, González Pérez and Popescu-Pampu.

Proposition 4.2.1. — *Let R be a noetherian regular local ring of dimension 2, which is a flat module over a field or a discrete valuation ring. Let Δ be any irreducible curve in $\operatorname{Spec} R$. Then the function*

$$d_\Delta(D_1,D_2):=\begin{cases}(D_1\cdot\Delta)(D_2\cdot\Delta)/(D_1\cdot D_2) & \text{if } D_1\neq D_2\\ 0 & \text{if } D_1=D_2\end{cases}$$

is an ultrametric distance on the space of irreducible curves in $\operatorname{Spec} R$ different from Δ.

Proof. — For those rings R which further satisfy the property of containing $\mathbf{C}$, this is proved in [GBGPPP18]. The proof only relies on (i) the existence of embedded resolutions of divisors in the spectra of such rings, and (ii) the negativity of the intersection matrix of the exceptional divisor of a projective birational morphism between spectra of such rings. Both results still hold under our weaker assumptions: see [Liu02, Theorem 9.2.26] for (i) and [Liu02, Theorem 9.1.27, Remark 9.1.28] for (ii). □

Arithmetic intersection multiplicities. — Suppose now that $\mathscr{X}$ is a 2-dimensional regular Noetherian scheme, proper flat over a discrete valuation ring $\mathscr{O}$. A divisor on $\mathscr{X}$ is called *horizontal* (respectively vertical) if each of the irreducible components of the support $|D|$ is flat over $\mathscr{O}$ (respectively contained in the special fibre $\mathscr{X}_\kappa$). We extend $(\,\cdot\,):=(\,\cdot\,)_{\mathscr{X}}$ to a bilinear form $(\,\bullet\,)$ on pairs of divisors on $\mathscr{X}$ sharing no common horizontal irreducible component of the support by

$$(\mathscr{X}_k\bullet V):=0$$

if V is any vertical divisor.

Denote by X the generic fibre of $\mathscr{X}$. If $D\in \operatorname{Div}^0(X)$ with Zariski closure $\overline{D}$ in $\mathscr{X}$, a *flat extension* of D is a divisor $\widehat{D}\in\operatorname{Div}(\mathscr{X})_{\mathbf{Q}}$ such that $\widehat{D}-\overline{D}$ is vertical and

$$(\widehat{D}\bullet V)=0$$

for any vertical divisor V on $\mathscr{X}$. A flat extension of D exists and it is unique up to addition of rational linear combinations of the connected components of $\mathscr{X}_k$.

The arithmetic intersection multiplicity on divisors with disjoint supports in $\operatorname{Div}^0(X)$ is then defined by

$$m_X(D_1,D_2):=(\overline{D}_1\bullet\widehat{D}_2)=(\widehat{D}_1\bullet\overline{D}_2)\in\mathbf{Q}.$$

4.3. Decay of intersection multiplicities. — We continue using the notation introduced in §4.1. Let $m_{\overline{w}}:=m_{X_{H_{0,\overline{w}}}}$. Developing the approximation argument sketched in the introduction, will show that for any degree-0 divisor D on X_{H_0}, we have

$$m_{\overline{w}}(Z_{a\varpi^s}[1]_U,D)=O(q_F^s)$$

in L, uniformly in a.

Let $U_{0,v}:=\mathrm{GL}_2(\mathscr{O}_{F,v})\subset\mathbf{B}_v^\times$, let $X_0:=X_{U^vU_{0,v}}$, let $\mathscr{X}_0$ be the canonical model of X_{0,F_v} over $\mathscr{O}_{F,v}$, and let $\mathscr{X}_0'$ be its base-change to $\mathscr{O}_{H_{0,\overline{w}}}$, which is smooth over $\mathscr{O}_{H_{0,\overline{w}}}$ (see [Car86]). Let $\mathscr{X}$ be the integral closure of $X_{H_{0,\overline{w}}}$ in $\mathscr{X}_0'$, which is a regular model of $X_{H_{0,\overline{w}}}$ over the ring of integers $\mathscr{O}_{H_{0,\overline{w}}}$, and let $\mathrm{p}\colon\mathscr{X}\to\mathscr{X}_0'$ be the projection. We denote by κ the residue field of $H_{0,\overline{w}}$.

Lemma 4.3.1. — *Let $z_s \in X_{H_0}$ be a CM point with pseudo-conductor s, let $\overline{z}_s$ be its closure in $\mathscr{X}$, and let $y \in \mathscr{X}_\kappa$ be its reduction modulo $\overline{w}$. Let $\mathscr{X}_y = \operatorname{Spec} \mathscr{O}_{\mathscr{X},y}$, let $\mathscr{X}_{y,\kappa}$ denote its special fibre. Then*

$$(\overline{z}_s \cdot [\mathscr{X}_{y,\kappa}])_{\mathscr{X}_y} = [\kappa(y) : \kappa] q_{F,v}^s.$$

Proof. — We will deduce this from Gross's theory of quasicanonical liftings [Gro86], which we recall. The situation is purely local and we drop all subscripts v and $\overline{w}$. For $E \subset K \subset E^{\rm ab}$, let $K^{\rm un}$ be the maximal unramified extension of K contained in $E^{\rm ab}$. Let k be the residue field of $\mathscr{O}_{E^{\rm un}}$. By [Car86, §7.4], for any supersingular point $y_0 \in \mathscr{X}_{0,\mathscr{O}_E^{\rm un}}$, the completed local ring of $\mathscr{X}_{0,\mathscr{O}_E^{\rm un}}$ at y_0 is isomorphic to $\mathscr{O}_{E^{\rm un}}[\![u]\!]$ and it is the deformation ring of formal modules studied by Gross. The main result of [Gro86] is that for any CM point $z_0 \in X_{0,E^{\rm un}}$, there exist a unique integer t (the *conductor* of z_0) such that the following hold. First, the field $E^{\rm un}(z_0)$ is the abelian extension $E^{(t)}$ of $E^{\rm un}$ with norm group $(\mathscr{O}_F + \varpi^t \mathscr{O}_E)^\times / \mathscr{O}_F^\times$, which is totally ramified of some degree d_t. Secondly, the inclusion of the Zariski closure $\overline{z}_0 \hookrightarrow \mathscr{X}_{0,\mathscr{O}_E^{\rm un}}$ gives rise to a map of complete local rings

$$\mathscr{O}_{E^{\rm un}}[\![u]\!] \to \mathscr{O}_{E^{(t)}} = \mathscr{O}(\overline{z}_0)$$

which sends u to a uniformiser $\varpi^{(t)}$ of $E^{(t)}$. It follows that if μ_t is the minimal polynomial of $\varpi^{(t)}$,

$$(\overline{z_0} \cdot \mathscr{X}_{0,k}) = \dim_k \mathscr{O}_{E^{\rm un}}[\![u]\!]/(\varpi_E, \mu_t(u)) = \dim_k \mathscr{O}_{E^{(t)}}/\varpi_E = d_t.$$

Consider now the situation of the lemma. By the projection formula,

$$(\overline{z}_s \cdot [\mathscr{X}_{y,\kappa}])_{\mathscr{X}_y} = (\overline{z}_s \cdot {\rm p}^* \mathscr{X}_{0,\kappa_0})_{\mathscr{X}_y} = ({\rm p}_* \overline{z}_s \cdot \mathscr{X}_{0,\kappa})_{\mathscr{X}'_{0,y'}} = [H_0(z_s) \colon H_0(z_{0,s})] \cdot (\overline{z}_{0,s} \cdot \mathscr{X}_{0,\kappa})_{\mathscr{X}'_{0,y'}}$$

where $y' = {\rm p}(y)$ and $z_{0,s} = {\rm p}(z_s) \in X_{0,H_0}$ is a CM point. The last intersection multiplicity is $[\kappa(y) : \kappa]$ times the multiplicities of the base-changed divisors to the ring of integers of $H_0^{\rm un}$, where $z_{0,s}$ remains an irreducible divisor since $H_0(z_{0,s}) \subset H_0(z_s)$ is totally ramified over H_0. We perform such base-change to $\mathscr{O}_{H_0^{\rm un}}$ without altering the notation. Let z be the image of $z_{0,s} \in X_{H_0^{\rm un}}$ in $X_{E^{\rm un}}$, and let t be the conductor of z; so $E^{(t)} \subset H_0^{\rm un}(z_{0,s})$. The fibre above $z \cong \operatorname{Spec} E^{(t)}$ in $X_{H_0^{\rm un}}$ is

$$\operatorname{Spec} E^{(t)} \otimes_{E^{\rm un}} H_0^{\rm un} = \operatorname{Spec} H_0^{\rm un}(z_{0,s})^{\oplus c}$$

for $c = [E^{(t)} : E^{\rm un}] \cdot [H_0^{\rm un} : E^{\rm un}] / [H_0(z_{0,s}) : H_0]$, and $z_{0,s}$ is one of the factors in the right hand side. By the projection formula applied to $\mathscr{X}_0 \times \operatorname{Spec} \mathscr{O}_{H_0^{\rm un}} \to \mathscr{X}_0 \times \operatorname{Spec} \mathscr{O}_{E^{\rm un}}$, we have

$$\begin{aligned}[H_0(z_s) \colon H_0(z_{0,s})] \cdot (\overline{z}_{0,s} \cdot \mathscr{X}_{0,\kappa})_{\mathscr{X}'_{0,y'}} &= [\kappa(y) : \kappa] c^{-1} \cdot [H_0(z_s) \colon H_0(z_{0,s})] \cdot [H_0^{\rm un} : E^{\rm un}] \cdot d_t \\ &= [\kappa(y) : \kappa] \cdot [H_0(z_s) : H_0] \cdot d_t^{-1} \cdot d_t = [\kappa(y) : \kappa] q_F^s.\end{aligned}$$

□

Proposition 4.3.2. — *Let D be a divisor in $\mathscr{X}$. There exists an integer N such that for all $s \geq N$ the following holds.*

Let $z_s \in X_{H_0}$ be a CM point of pseudo-conductor s, and let $\overline{z_s}$ be its closure in $\mathscr{X}$. Then there exist a unique irreducible component $V \subset \mathscr{X}_\kappa$ and a positive constant $\rho \in \mathbf{Q}^\times$, such that:

- *the denominator of ρ is bounded independently of z_s;*
- *if we write*

$$D = c\mathscr{X}_\kappa + D'$$

with $c \in \mathbf{Q}$ and $D' \in \operatorname{Div}(\mathscr{X})_{\mathbf{Q}}$ a divisor whose support does not contain V, then

$$(\overline{z}_s \cdot D) = c[\kappa(y) : \kappa] q_{F,v}^s + \rho(V \bullet D). \quad (4.3.1)$$

Proof. — By linearity and Lemma 4.3.1 it suffices to prove the first formula for an irreducible divisor $D \neq V$ (once found which V is to be chosen). We will omit all subscripts v, $\overline{w}$ and use some of the notation introduced in the proof of Lemma 4.3.1.

Let $y \in \mathscr{X}_\kappa$, let Δ be any irreducible horizontal divisor in $\mathscr{X}_y$. We claim that *the intersection multiplicities*

$$(\Delta \cdot z_s),$$

are bounded by an absolute constant as z_s ranges among CM points of sufficiently large pseudo-conductor reducing to y. Indeed $(\Delta \cdot z_s)$ is bounded by the degree of $\mathrm{q}\colon \mathscr{X} \to \mathscr{X}_0$ near y, times the intersection multiplicities of the pushforward divisors to $\mathscr{X}_{0,\mathscr{O}_{E,v}}$. Similarly to the proof of Lemma 4.3.1 we may estimate this intersection in the base change of $\mathscr{X}_0$ to $\mathscr{O}_E^{\mathrm{un}}$. The base-change of the divisor $\mathrm{q}_* z_s$ equals a sum of CM points of $\mathscr{X}_0 \times_{\mathrm{Spec}\,\mathscr{O}_F} \mathrm{Spec}\,\mathscr{O}_{E^{\mathrm{un}}}$; because H_s/H_0 is totally ramified, the number of points in this divisor is bounded by an absolute constant and the conductors of all those CM points go to infinity with s. Thus it suffices to show that if Δ_0 is a fixed horizontal divisor on $\mathscr{X}_0 \times_{\mathrm{Spec}\,\mathscr{O}_F} \mathrm{Spec}\,\mathscr{O}_{E^{\mathrm{un}}}$, its intersection multiplicity with CM points of conductor t is bounded as $t \to \infty$.

Let $z \in \mathscr{X}_{0,\mathscr{O}_E^{\mathrm{un}}}$ be a CM point of conductor t, and let $y_0 \in \mathscr{X}_{0,k}$ be the image of the reduction of t. Now write the image of Δ_0 in the completion of $\mathscr{X}_{0,\mathscr{O}_{E^{\mathrm{un}}}}$ at y_0 as

$$\widehat{\Delta}_0 = \mathrm{Spec}\,\mathscr{O}_{E^{\mathrm{un}}}[\![u]\!]/(f) \subset \mathrm{Spec}\,\mathscr{O}_{E^{\mathrm{un}}}[\![u]\!],$$

with $f = \sum_{i=1}^d a_i u^i$ an integral non-constant monic polynomial. Let $\widetilde{f} \in k[u]$ be the reduction of f. Then

$$(\Delta_0 \cdot \overline{z}) = \dim_k \mathscr{O}_{E^{\mathrm{un}}}[\![u]\!]/(f, \mu_t) = \dim_k \mathscr{O}_{E^{(t)}}/(f(\varpi^{(t)})) = \dim_k \mathscr{O}_{E^{(t)}}/((\varpi^{(t)})^{\deg(\tilde{f})}) = \deg(\tilde{f}) \leq d$$

if t is sufficiently large, since the normalised valuations of $\varpi^{(t)}$ decrease to 0 as $t \to \infty$. This completes the proof of the claim.

Write $[\mathscr{X}_{y,\kappa}] = \sum_{V'} e_{V'} V'$, and consider the ultrametric distance d_Δ of Proposition 4.2.1 (applied to $R = \mathscr{O}_{\mathscr{X},y}$). As

$$\sum_{V'} \frac{e_{V'}}{(\overline{z}_s \cdot \Delta)(V' \cdot \Delta)} d_\Delta(\overline{z}_s, V')^{-1} = (\overline{z}_s \cdot [\mathscr{X}_{y,\kappa}]) = [\kappa(y)\colon \kappa] q_F^s$$

goes to infinity with s and the coefficients of the $d_\Delta(\overline{z}_s, V')^{-1}$ are bounded, the minimum of the $d_\Delta(\overline{z}_s, V')$ goes to 0 with s. Let $V \subset \mathscr{X}_{y,\kappa}$ be a vertical component realising the minimum. Then if s is sufficiently large and $D \neq V$ is irreducible with reduction y, we have $d_\Delta(\overline{z}_s, V) < d_\Delta(V, D)$, so that by Proposition 4.2.1,

$$d_\Delta(\overline{z}_s, D) = d_\Delta(V, D).$$

Unwinding the definitions

$$(\overline{z}_s \cdot D) = \rho(V \cdot D), \qquad \rho = \frac{(\overline{z}_s \cdot \Delta)}{(V \cdot \Delta)},$$

so that the denominator of ρ is bounded by $\max_V\{(V \cdot \Delta)\}$. Applied to a vertical $D = V' \neq V$, the formula shows the uniqueness of V for large s. □

Corollary 4.3.3. — *If $D \in \mathrm{Div}^0(X_{H_0})_L$ is any degree-zero divisor, then for all sufficiently large s and all a,*

$$m_{\overline{w}}(Z_{a\varpi^s}(\phi^\infty)[1]_U, D) = O(q_{F,v}^s)$$

in L, where the implied constant can be fixed independently of a and s.

Proof. — Let $\widehat{D}$ be a flat extension of D to a divisor on $\mathscr{X}$ (with coefficients in L), and let $\overline{Z}_{a,s}$ be the Zariski closure of $Z_{a\varpi^s}(\phi^\infty)[1]_U$. Then by Propositions 4.1.4 and 4.3.2,

$$m_{\overline{w}}(Z_{a\varpi^s}(\phi^\infty)[1]_U, D) = (\overline{Z}_{a,s} \cdot \widehat{D}) = Aq_F^s + \sum_i \lambda_i(V_i \bullet \widehat{D})$$

for some vertical components V_i and $A, \lambda_i \in L$ whose denominators are bounded independently of a. (For λ_i, the boundedness is part of Proposition 4.1.4; for A, it holds because each c in (4.3.1) is.) By definition of flat divisor, $(V_i \bullet \widehat{D}) = 0$ for all i. □

4.4. Decay of local heights. — Recall that we need to prove (Proposition 3.6.3) that

$$T_{\iota_{\mathfrak{p}}}(\sigma^\vee)\widetilde{Z}(\phi^\infty, \chi)(v)$$

is a v-critical element of $\overline{\mathbf{S}}'$. As in [I, §9.2, proof of Proposition 9.2.1], this follows if we show that for all $\overline{w}|v$ and all $a \in \mathbf{A}^{S_1\infty,\times}$ with $v(a) = r_v$,

$$\langle \widetilde{Z}_{a\varpi^s}(\phi^\infty)_U[1], \mathrm{T}_{\iota_{\mathfrak{p}}}(\sigma^\vee)_U^{\mathrm{t}} t_\chi \rangle_{\overline{w}} = O(q_{F,v}^s) \tag{4.4.1}$$

in $\Gamma_F \hat{\otimes} L(\chi)$. (Here $\mathrm{T}_{\iota_{\mathfrak{p}}}(\sigma^\vee)_U$ is a certain correspondence.) By using, as in *loc. cit.*,[(6)] the integrality result of Proposition 4.2.3 *ibid.*, the left hand side of (4.4.1) can be expressed as

$$c\varpi_{L(\chi)}^{-(d_0+d_{1,s}+d_2)} N_{H_{0,\overline{w}}/F_v} \circ N_s(h_s), \qquad h_s \in H_{s,\overline{w}}^\times \hat{\otimes} \mathscr{O}_{L(\chi)}, \tag{4.4.2}$$

where $c \in \mathscr{O}_{L(\chi)}$, $\varpi_{L(\chi)}$ is a uniformiser in $L(\chi)$, N_s is the norm from $H_{s,\overline{w}}$ to $H_{0,\overline{w}}$, and the integers d_0, $d_{1,s}$, d_2 are as recalled before Lemma 4.4.2 below, where we show they are bounded independently of s. Thus it suffices to show that, for some constant c',

$$N_s(h_s) \equiv 0 \qquad \text{in } H_{0,\overline{w}}^\times \otimes \mathscr{O}_{L(\chi)}/q^{s-c'}\mathscr{O}_{L(\chi)}. \tag{4.4.3}$$

Lemma 4.4.1. — *For all $s' \leq s$, the restriction of the $\overline{w}$-adic valuation yields an isomorphism of $\mathscr{O}_{L(\chi)}$-modules*

$$N_s(H_{s,\overline{w}}^\times \hat{\otimes} \mathscr{O}_{L(\chi)})/q^{s'} \cdot (H_{0,\overline{w}}^\times \hat{\otimes} \mathscr{O}_{L(\chi)}) \to \mathscr{O}_{L(\chi)}/q^{s'}\mathscr{O}_{L(\chi)}.$$

Proof. — We drop all subscripts $\overline{w}$. Recall that the extension H_s/H_0 is totally ramified of degree q^s. Let $\varpi_s \in \mathscr{O}_{H_s}$ be a uniformiser; then $\omega_0 := N_s(\varpi_s)$ is a uniformiser of H_0. For $* = 0, s$ we have the decompositions

$$H_*^\times \hat{\otimes} \mathscr{O}_{L(\chi)} = \mathscr{O}_{H_*}^\times \hat{\otimes} \mathscr{O}_{L(\chi)} \oplus \varpi_* \otimes \mathscr{O}_{L(\chi)}.$$

The map N_s respects the decompositions and, by local class field theory, has image

$$q^s \cdot (\mathscr{O}_{H_0}^\times \hat{\otimes} \mathscr{O}_{L(\chi)}) \oplus \varpi_0 \otimes \mathscr{O}_{L(\chi)}.$$

The valuation map annihilates the first summand and sends the second one isomorphically to $\mathscr{O}_{L(\chi)}$. The result follows. □

Proof of Proposition 3.6.3. — By the discussion in the present subsection, we are reduced to showing (4.4.3); equivalently, thanks to the previous lemma, that

$$\overline{w}(N_s(h_s)) \equiv 0 \pmod{q^{s-c'}\mathscr{O}_{L(\chi)}}.$$

By the construction of h_s and [I, Proposition 4.3.1] applied to the curve X_{U/H_0}, we have

$$\overline{w}(N_s(h_s)) = c\varpi_{L(\chi)}^{-(d_0+d_{1,s}+d_2)} m(\widetilde{Z}_{a\varpi^s}(\phi^\infty)_U[1], \mathrm{T}_{\iota_{\mathfrak{p}}}(\sigma^\vee)_U^{\mathrm{t}} t_\chi). \tag{4.4.4}$$

(6) Our field H_s should be compared to the H_s' of *loc. cit.*

By Corollary 4.3.3 applied to $D = t_\chi$, together with the next lemma proving the boundedness of $d_{1,s}$, the right-hand side of (4.4.4) is $O(q_{F,v}^s)$. □

Finally, we take care of the sequence

$$d_{1,s} := \mathrm{length}_{\mathscr{O}_{L'}} H^1(H_{0,\overline{w}}, T_s''^*(1) \otimes L'/\mathscr{O}_{L'})_{\mathrm{tors}},$$

where $T_s'' \subset T_pJ_U$ is the same sub-$\mathscr{O}_L[\mathscr{G}_F]$-module as in [I, Proof of Proposition 9.2.3], and $L' = L(\chi)$. As

$$H^1(H_{0,\overline{w}}, T_s''^*(1)\otimes L'/\mathscr{O}_{L'})_{\mathrm{tors}} \cong H^0(H_{0,\overline{w}}, T_s''^*(1)) \subset H^0\left(E_w, T_pJ_U^*(1)_{|\mathscr{G}_{E_w}} \otimes \mathscr{O}_{L'}[\mathrm{Gal}(H_{s,\overline{w}}/E_w)]\right),$$

the boundedness follows from the next lemma.

Lemma 4.4.2. — *Let $\Gamma_\infty := \mathrm{Gal}(H_{\infty,\overline{w}}/E_w) \cong E^\times\backslash E^\times_{\mathbf{A}^\infty}/U^vU^\circ_{F,v}$. We have*

$$H^0(E_w, T_pJ_U^*(1) \otimes_{\mathscr{O}_L} \mathscr{O}_{L'}[\![\Gamma_\infty]\!]) = 0.$$

Proof. — The proof is largely similar to that of [I, Lemma 9.2.4], to which we refer for more details. It suffices to show that for all Hodge–Tate characters $\psi\colon \mathscr{G}_{E_w} \to L'^\times$ factoring through Γ_∞, the space

$$H^0(E_w, V_pJ_U(\psi)) = \mathbf{D}_{\mathrm{crys}}(V_pJ_U(\psi))^{\varphi=1}$$

is zero. Here φ is the crystalline Frobenius, and we may prove the same result after replacing it with φ^d , $d = [E_w : E_{w,0}]$ where $E_{w,0}$ the maximal unramified extension of $\mathbf{Q}_p$ contained in E_w. Since φ^d acts with negative weights on $\mathbf{D}_{\mathrm{crys}}(V_pJ_U)$, it suffices to show that it acts with weight 0 on $\mathbf{D}_{\mathrm{crys}}(\psi^m)$ for m such that ψ^m is crystalline.

Since ψ is trivial on U_F°, the Hodge–Tate weights $(n_\tau)_{\tau\in\mathrm{Hom}\,(E_w,\overline{L})}$ satisfy $n_\tau + n_{\tau c} = 0$ where c is the complex conjugation of E_w/F_v. The action of φ^d on $\mathbf{D}_{\mathrm{crys}}(\psi^m)$ is by

$$\psi \circ \mathrm{rec}_{E,w}(\varpi_w)^{-m} \cdot \prod_{\tau\in\mathrm{Hom}\,(E_w,\overline{L})} \varpi_w^{mn_\tau}, \tag{4.4.5}$$

where $\varpi_w \in E_w$ is any uniformiser. Choose ϖ_w so that $\varpi_w^{e(E_w/F_v)} = \varpi_v$ is a uniformiser in F_v. Then $\varpi_w^c = \pm\varpi_w$, so that the second factor in (4.4.5) is ±1. On the other hand, the subgroup $F^\times\backslash F^\times_{\mathbf{A}^\infty}/U^\circ_{F,v}(U^v \cap F^\times_{\mathbf{A}^\infty}) \subset \Gamma_\infty$ is finite, hence ϖ_v and ϖ_w have finite order in Γ_∞. It follows that the first factor in (4.4.5) is a root of unity too, hence φ^d acts with weight 0 on $\mathbf{D}_{\mathrm{crys}}(\psi^m)$. □

Summary. — We have just completed the proof of Proposition 3.6.3. It implies Proposition 3.6.2, which together with Theorem 3.6.1 implies the kernel identity of Theorem 3.4.3. By Corollary 3.4.4, that implies Theorem 2.2.1, which in turn is equivalent to Theorem B by Lemma 2.2.2.

Appendix A. Local integrals

Throughout this appendix, unless otherwise noted v denotes a place of F above p. We use some of the notation introduced in §3.1, in particular the Weil representation r (see [I, §3.1] or [YZZ12] for the formulas defining it).

A.1. Gamma factors. — We study local gamma factors, in order to relate the interpolation factors of the$p$-adic L-function used here and those of the one from [I].

Lemma A.1.1. — *Let $\xi\colon E_w^\times \to \mathbf{C}^\times$ and $\psi\colon E_w^\times \to \mathbf{C}^\times$ be characters, with $\psi \neq \mathbf{1}$. Let dt be a Haar measure on $E_v^\times$. Then*

$$\int_{E_w^\times} \xi(t)\psi(t)dt = \frac{dt}{d_\psi t} \cdot \xi(-1) \cdot \gamma(\xi,\psi)^{-1}.$$

The left-hand side is to be understood in the sense of analytic continuation from characters $\xi|\cdot|^s$ for $\Re(s) \gg 0$.

Proof. — We may fix $dt = d_\psi t$. Then the result follows from the functional equation for GL_1 ([BH06, (23.4.4)]):

$$Z(\phi, \xi) = \gamma(\xi, \psi)^{-1} Z(\hat{\phi}, \xi^{-1}|\ |), \tag{A.1.1}$$

where for a Schwartz function ϕ on E_w,

$$Z(\phi, \xi) := \int_{E_w^\times} \phi(t)\xi(t) d_\psi t, \qquad \hat{\phi}(t) := \int_{E_w} \phi(x)\psi(xt) d_\psi x.$$

Namely, we insert in (A.1.1) the function $\hat{\phi} := \delta_{-1+\varpi_v^n \mathscr{O}_F} := \mathrm{vol}(1+\varpi_v^n \mathscr{O}_{F,v}, d_\psi t)^{-1} \mathbf{1}_{-1+\varpi_v{}^n \mathscr{O}_{F,v}}$, $n \geq 1$, approximating a delta function at $t = -1$. Then by Fourier inversion $\phi(t) = \hat{\hat{\phi}}(-t) = \delta_{1+\varpi_v^n \mathscr{O}_F} * \psi(t)$, which if n is sufficiently large (depending on the conductor of ξ) has the same integral against ξ as $\psi(t)$. □

In what follows, by a 'Weil–Deligne representation of (the Weil group) of E_v' we mean the data, for each place $w|v$ of E, of a Weil–Deligne representation of the Weil group of E_w. To such an object we attach additively L-, γ-, and ε-factors. The induction functor $\mathrm{Ind}_{E_v}^{F_v}$ from Weil–Deligne representations of E_v to Weil–Deligne representations of F_v is defined when $E_v = F_v \oplus F_v$ too.

Lemma A.1.2. — *Let W be a Weil–Deligne representation of the Weil group of F_v and let $\psi\colon F_v \to \mathbf{C}^\times$ be a nontrivial character. Then for any $a \in F_v^\times$,*

$$\gamma(W, a.\psi)^{-1} = \det W(a) \cdot \gamma(W, \psi)^{-1}.$$

Proof. — This follows from the corresponding result for ε-factors, proved in [Tat79]. □

For the rest of this appendix, we denote $\psi_{E,v} = \psi_v \circ \mathrm{Tr}_{E_v/F_v}$.

Lemma A.1.3. — *Let W be a Weil–Deligne representation of the Weil group of E_v. Then*

$$C'_v(W_v, \psi_v) := \gamma(W, \psi_{E,v})/\gamma(\mathrm{Ind}_{E_v}^{F_v}(W, \psi_v))$$

depends only on ψ_v and $\dim W$. It satisfies $C'_v(a.\psi_v) = \eta_v^{\dim W}(a) C'_v(\psi_v)$.

Proof. — As L-factors are inductive and independent of ψ_v we may replace γ- by ε-factors in the statement. The ratio $C_v(W_v, \psi_v)$ descends to a function on the Grothendieck group R of virtual Weil–Deligne representations. By [Tat79], this group is generated by the trivial representation and representations of degree 0, and the ratio is 1 for representations of degreee 0 (inductivity in degree 0 of ε-factors). We deduce that $C(W_v, \psi_v)$ equals $\varepsilon(\eta_v, \psi_v)^{\dim W_v}$ up to a function depending only on $\dim W$ and not on ψ_v. □

Proposition A.1.4. — *The ratio $C(\chi'_p \psi_p)$ defined in* (2.1.3) *is independent of χ'_p. It defines an element of $\mathscr{O}(\Psi_p, \omega_p \alpha_p^{-2}|\cdot|_p^{-2})$.*

Proof. — By the definition of $e_p(V_{(A,\chi')}, \psi_p)$ and a comparison of [I, Lemma A.1.1] with Lemma A.1.1 applied to $\prod_{w|v} \chi'_w \alpha_v |\cdot|_v \circ q_w$, we have

$$C(\chi'_p \psi_p) = \prod_{v|p} \frac{C'_v(W_v^+, \psi_v)}{\gamma(\mathrm{ad}(W_v(1)^{++}), \psi_v)},$$

where $\psi_p = \prod_{v|p} \psi_v$ and $\mathrm{ad}(W_v(1)^{++})$ is defined above Theorem A. Then the independence of χ' follows from the first part of Lemma A.1.3. The equivariance property follows from the second part of the latter, together with Lemma A.1.2 and the computation

$$\det(W_v^+ \otimes \mathrm{Ind}_{E_v}^{F_v} \chi') = \eta_v \alpha_v^2 |\cdot|_v^2 \chi'_{|F_v^\times}.$$

□

A.2. Toric period at p. — We compare the toric period at a p-adic place with the interpolation factor. Denote by $P_v \subset \mathrm{GL}_2(F_v)$ the upper triangular Borel subgroup.

Lemma A.2.1. — *The quotient space $K_1^1(\varpi^{r'})\backslash \mathrm{GL}_2(F,v)/P_v$ admits the set of representatives*

$$n^-(c) := \begin{cases} \left(\begin{smallmatrix} 1 & \\ c & 1\end{smallmatrix}\right) & \text{if } c \neq \infty \\ \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) & \text{if } c = \infty, \end{cases} \qquad c \in \mathscr{O}_{F,v}/\varpi^{r'}\mathscr{O}_{F,v} \cup \{\infty\}.$$

Proposition A.2.2. — *Let $\chi \in \mathscr{Y}^{\mathrm{l.c.}}$ be a finite-order character, let r be sufficiently large (that is, satisfying the v-component of Assumption 3.4.2), and let W_v be as in* (3.1.1), *$\phi_v = \phi_{v,r}$ be as in* (3.1.4).

Let $\pi_v = \sigma_v$ and let $(,)_v \colon \pi_v \times \pi_v^\vee \to L$ be a duality pairing satisfying the compatibility of [I, (5.1.2)] *with the local Shimizu lift.*[(7)] *Finally, let $Z_v^\circ(\alpha_v, \chi_v)$ be the interpolation factor of the p-adic L-function of* [I, Theorem A].

Then for all sufficiently large $r' > r$,

$$Q_{(,),v,d^\circ t_v}(\theta_v(W_v, \alpha|\cdot|_v(\varpi_v)^{-r'_v} w_{r',v}^{-1}\phi_v), \chi_v) = |d|_v^2 |D|_v \cdot L(1,\eta_v)^{-1} \cdot Z_v^\circ(\alpha_v, \chi_v),$$

where $Q_{(,),v,d^\circ t_v}$ uses the measure $d^\circ t_v = |d|_v^{-1/2}|D|_v^{-1/2} dt$.

Proof. — We drop all subscripts v, and assume as usual that ψ has level d^{-1}. Let

$$\text{(A.2.1)} \qquad Q^\sharp_{(,)}(f_1, f_2, \chi_v) = \int_{E^\times/F^\times} \chi(t)(\pi(t)f_1, f_2)\, dt,$$

where dt is the usual Haar measure on $E^\times/F^\times$, giving volume $|d|^{1/2}|D|^{1/2}$ to $\mathscr{O}_E^\times/\mathscr{O}_F^\times$.

By the definitions and [I, Lemma A.1.1] (which expresses Z° as a normalised integral), it suffices to show that

$$Q^\sharp(\theta(W, \alpha|\cdot|(\varpi)^{-r'} w_{r'}^{-1}\phi), \chi) = |d|^{3/2}|D| \cdot L(1,\eta)^{-1} \cdot \int_{E^\times} \alpha|\cdot| \circ q(t)\chi(t)\psi_E(t)\, dt.$$

By [I, Lemma 5.1.1] (which spells out a consequence of the normalisation of the local Shimizu lifting) and Lemma A.2.1 we can write

$$Q^\sharp := Q(\theta(W, \alpha|\cdot|(\varpi)^{-r'} w_{r'}^{-1}\phi_r), \chi_v) = \sum_{c \in \mathbf{P}^1(\mathscr{O}_F/\varpi^{r'})} Q^{\sharp\,(c)}$$

where for each c,

$$Q^{\sharp\,(c)} := |d|^{-3/2} \cdot \alpha|\,|(\varpi)^{-r} \int_{F^\times} W(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right) n^-(c))$$
$$\int_{T(F)} \chi(t) \int_{P(\varpi^{r'})\backslash K_1^1(\varpi^{r'})} |y| r(n^-(c)k w_r^{-1})\phi(yt^{-1}, y^{-1}q(t)) dk\, dt\, \frac{d^\times y}{|y|}.$$

Here $P(\varpi^{r'}) = P \cap K_1^1(\varpi^{r'})$.

It is easy to see that $Q^{\sharp(\infty)} = 0$ (observe that $\phi_{2,r}(0) = 0$). For $c \neq \infty$, we have

$$n^-(c) w_{r'}^{-1} = w_{r'}^{-1} \begin{pmatrix} 1 & -c\varpi^{-r'} \\ & 1 \end{pmatrix},$$

(7) In *loc. cit.*, the pairing $(,)_v$ is denoted by $\mathscr{F}_v$.

and when $x = (x_1, x_2)$ with $x_2 = 0$:

$$r(n^-(c)w_{r'}^{-1})\phi(x,u) = \int_{\mathbf{V}} \psi_E(ux_1\xi_1)\psi(-ucq(\xi))\phi_r(\xi, \varpi^{r'}u)d\,\xi$$

On the support of the integrand we have $v(u) = \varpi^{-r'}$ and $v(q(\xi)) \geq r$, by the definition of ϕ_r. If $v(c) < r' - r - v(d)$, the integration in $d\xi_2$ gives 0; hence $Q^{\sharp(c)} = 0$ in that case.

Suppose from now on that $v(c) \geq r' - r - v(d)$. Then $\psi(-ucq(\xi)) = 1$ and

$$r(n^-(c)w_{r'}^{-1})\phi(x,u) = |d|^3|D|L(1,\eta)^{-1}|\varpi|^r\psi_{E,r}(\varpi^{-r'}x_1)\delta_{1,U_F,r}(\varpi^r u).$$

where $\psi_{E,r} := \mathrm{vol}(\mathscr{O}_E)^{-1}\cdot\delta_{1,U_{T,r}} * \psi_E$, and we have noted that $\hat{\phi}_{2,r}(0) = e^{-1}|d|\cdot\mathrm{vol}(q^{-1}(-1+\varpi^r\mathscr{O}_F)\cap\mathscr{O}_{\mathbf{V}_2}) = |\varpi|^r|d|^3|D|^{1/2}L(1,\eta)^{-1}$.

If r' is sufficiently large, the Whittaker function W is invariant under $n^-(c)$. Then

$$Q^{\sharp\,(c)} = |d|^{3/2}|D|L(1,\eta)^{-1}\cdot|\varpi|^{r-r'}\alpha(\varpi)^{-r'}\cdot|d|^{1/2}\zeta_{F,v}(1)^{-1}$$
$$\cdot\int_{F^\times} W(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))\int_{T(F)}\chi(t)\psi_{E,r}(\varpi^{-r'}x_1)\delta_{1,U_{F,r}}(\varpi^{r'}y^{-1}q(t))dtd^\times y,$$

where $|\varpi|^{-r'}|d|^{1/2}\zeta_{F,v}(1)^{-1}$ appears as $\mathrm{vol}(P(\varpi^{r'})\backslash K_1^1(\varpi^{r'}))$.

Integrating in $d^\times y$ and summing the above over the $q_F^{r+v(d)} = |d|^{-1}|\varpi|^{-r}$ contributing values of c, we find

$$Q^\sharp = |d|^{3/2}|D|\cdot L(1,\eta)^{-1}\cdot\int_{E^\times}\chi(t)\alpha|\ |\circ q(t)\psi_E(t)\,dt,$$

as desired. □

Corollary A.2.3. — *If χ_p is not exceptional, then the quaternion algebra $\mathbf{B}$ over $\mathbf{A}$ satisfying $\mathrm{H}(\pi_{\mathbf{B}},\chi)\neq 0$ is indefinite at all primes $v|p$.*

Proof. — By Propositions A.2.2 and A.1.4, if χ_p is not exceptional, then for all $v|p$ the functional $Q_v \in \mathrm{H}(\pi_{M_2(F_v)},\chi_v)\otimes\mathrm{H}(\pi^\vee_{M_2(F_v)},\chi_v^{-1})$ is not identically zero. □

Appendix B. Errata to [I]

The salient mistakes are the following two: the statement of the main theorem is off by a factor of 2; and the Schwartz function $\phi_{2,p}$ given by the local Siegel–Weil formula at p needs to be *different* from the Schwartz function used to construct the Eisenstein family.

References in *italics* are directed to [I].

– *Theorem A*. It should be

$$L_{p,\alpha}(\sigma_E) \in \mathscr{O}_{\mathscr{Y}'\times\Psi_p}(\mathscr{Y}', \omega_p^{-1}\alpha_p^2|\cdot|_p^2\chi_{F,\mathrm{univ},p})^{\mathrm{b}}.$$

instead of $\mathscr{O}_{\mathscr{Y}'\times\Psi_p}(\mathscr{Y}', \omega_p^{-1}\chi_{F,\mathrm{univ},p})^{\mathrm{b}}$.

– *Theorem B*. The constant factor should be c_E and not $c_E/2$ (the latter is, according to [I, (1.1.3)], the constant factor of the Gross–Zagier formula in archimedean coefficients). The mistake is introduced in the proof of [I, Proposition 5.4.3], see below.

The heuristic reason for the difference is that the direct analogue of $s\mapsto L(1/2+s,\sigma_E,\chi)$ is $\chi_F \mapsto L_p(\sigma_E)(\chi\cdot\chi_F\circ q)$ (where q is the adelisation of the norm from E to F), whose derivative at $\chi_F = \mathbf{1}$ is twice our $\mathrm{d}_F L_p(\sigma_E)(\chi)$ (because the tangent map to

$$\chi_F \mapsto \chi' = \chi\cdot\chi_F\circ q \mapsto \omega^{-1}\cdot\chi'_{|\mathbf{A}^\times}$$

is multiplication by 2).

– *Theorem C*. Similarly, the constant factor should be $c_E/2$ in part 3, and c_E in part 4.

– *§2.1.* The space of p-adic modular forms is the closure of $M_2(K^pK_1^1(p^\infty)_p)$, not $M_2(K^pK^1(p^\infty)_p)$.
– *Proposition 2.4.4.1.* The multiplier in *equation (2.4.3)* should be $\alpha|\cdot|(\varpi^{-r}) = \prod_{v|p}\alpha_v|\cdot|_v(\varpi_v^{-r})$, and the statement holds for forms in $M_2(K^pK_1^1(p^r)_p)$. Similarly, the definition of $R^\circ_{r,v}$ in *Proposition 3.5.1* should have an extra $|\varpi_v|^{-r}$. The result of *Proposition A.2.2*, as modified below, holds true for this definition of $R^\circ_{r,v}$ (in *loc. cit.*, a complementary mistake appears between the third-last and second-last displayed equations in the proof).
– *Proposition 3.2.3.2* should be corrected as follows: *Let $v|p$ and let $\phi_{2,v} = \phi^\circ_{2,v}$ be as in* (3.1.4) (of the present paper). *Then*

$$W^\circ_{a,r,v}(1,u,\chi_F) = \begin{cases} |d_v|^{3/2}|D_v|^{1/2}\chi_{F,v}(-1) & \textit{if } \ v(a)\geq 0 \textit{ and } \ v(u)=0 \\ 0 & \textit{otherwise.}\end{cases}$$

– *Equation (3.7.1)* The right-hand side should have an extra factor of $\prod_{v|p}|d|_v^2|D|_v$ owing to the correction to *Proposition A.2.2.*
– *Lemma 5.3.1*: the left-hand side of the last equation in the statement should be $\langle f_1'(P_1), f_2'(P_2)\rangle_{J,*}$.
– *Proof of Proposition 5.4.3.* The second-last displayed equation should have the factor of 2 on the right hand side, not the left hand side:

$$2\ell_{\varphi^p,\alpha}(\widetilde{Z}(\phi^\infty,\chi)) = |D_F|^{1/2}|D_E|^{1/2}L(1,\eta)\langle \mathrm{T}_{\mathrm{alg},\iota_\mathfrak{p}}(\theta_{\iota_\mathfrak{p}}(\varphi,\alpha(\varpi)^{-r}w_r^{-1}\phi))P_\chi, P_\chi^{-1}\rangle.$$

Then the argument shows that, first,

$$\langle \mathrm{T}_{\mathrm{alg},\iota_\mathfrak{p}}(f_1\otimes f_2)P_\chi, P_{\chi^{-1}}\rangle_J = \frac{\zeta_F^\infty(2)}{(\pi^2/2)^{[F:\mathbf{Q}]}|D_E|^{1/2}L(1,\eta)}\prod_{v|p} Z_v^\circ(\alpha_v,\chi_v)^{-1}\cdot \mathrm{d}_F L_{p,\alpha}(\sigma_{A,E})(\chi)\cdot Q(f_1,f_2,\chi)$$

(without an incorrect factor of 2 introduced in the denominator of the right-hand side of (5.4.1) there); and secondly, that the above equation is equivalent to Theorem B with the correction explained above.

Also, an extra factor $\prod_{v|p}|d|_v^2|D|_v$ should be inserted in the right-hand sides of the last and fourth-last displayed equation, cf. the corrections to *(3.7.1)*, *Proposition A.3.1.*
– *Proposition 7.1.1(b)*: should be replaced by Proposition 3.5.1(b-c) of the present paper.
– *§7.2, third paragraph*: the coefficient in the second displayed equation should have $|D_E|^{1/2}$, not $|D_{E/F}|^{1/2}$, in the denominator.
– *Lemma 9.1.1* is corrected by Lemma 4.1.1 of the present paper (this does not significantly affect the rest).
– *Lemma 9.1.5*: the extension H_∞ is *contained* in a relative Lubin–Tate extension. This is the only property used.
– *Lemma A.2.1*: the list of purported representatives is missing the element $n^-(\infty) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, cf. Lemma A.2.1 above.
– *Proposition A.2.2*: the statement should be

$$R^\natural_{r,v}(W_v,\phi_v,\chi_v'^{\,\iota}) = |d|_v^2|D|_v\cdot Z_v^\circ(\chi_v') := |d|_v^2|D|_v\cdot\frac{\zeta_{F,v}(2)L(1,\eta_v)^2}{L(1/2,\sigma^\iota_{E,v},\chi_v')}\prod_{w|v} Z_w(\chi_w').$$

The factor $|d|_v^2|D|_v$ missing from [I] should first appear in the right-hand side of the displayed formula for $r(w_r^{-1})\phi(x,u)$ in the middle of the proof.
– *Proposition A.3.1*: should be replaced by Proposition A.2.2 of the present paper.

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