

FUNDAMENTAL BOUNDED RESOLUTIONS AND QUASI- (DF) -SPACES

J. C. FERRANDO, S. GABRIYELIAN, AND J. KAÇKOL

ABSTRACT. We introduce a new class of locally convex spaces E , under the name quasi- (DF) -spaces, containing strictly the class of (DF) -spaces. A locally convex space E is called a quasi- (DF) -space if (i) E admits a fundamental bounded resolution, i.e. an $\mathbb{N}^{\mathbb{N}}$ -increasing family of bounded sets in E which swallows all bounded set in E , and (ii) E belongs to the class \mathfrak{G} (in sense of Cascales–Orihuela). The class of quasi- (DF) -spaces is closed under taking subspaces, countable direct sums and countable products. Every regular (LM) -space (particularly, every metrizable locally convex space) and its strong dual are quasi- (DF) -spaces. We prove that $C_p(X)$ has a fundamental bounded resolution iff $C_p(X)$ is a quasi- (DF) -space iff the strong dual of $C_p(X)$ is a quasi- (DF) -space iff X is countable. If X is a metrizable space, then $C_k(X)$ is a quasi- (DF) -space iff X is a Polish σ -compact space. We provide numerous concrete examples which in particular clarify differences between (DF) -spaces and quasi- (DF) -spaces.

1. INTRODUCTION

An important class of locally convex spaces (lcs, for short), the class of (DF) -space, was introduced by Grothendieck in [20], we refer also to monographs [21] or [31] for more details.

Definition 1.1. A locally convex space E is called a (DF) -space if

- (1) E has a fundamental sequence of bounded sets, and
- (2) E is \aleph_0 -quasibarrelled.

All countable inductive limits of normed spaces (hence all normed spaces) are (DF) -spaces. Less evident facts are that the strong dual of a metrizable lcs is a (DF) -space and the strong dual of a (DF) -space is a metrizable and complete lcs, see [31, Theorem 8.3.9, Proposition 8.3.7].

The concept of (DF) -spaces has been generalized by Ruess [33, 34] (under the name (gDF) -spaces) and has also intensively studied by Nouredine [29, 30] who called them D_b -spaces. Their papers provided also some information about spaces which in [21] were called (df) -spaces. Next Adasch and Ernst [1, 2] provided another line of research around (DF) -spaces in the setting of general topological vector spaces. Note however that all known generalizations of (DF) -spaces E kept up the condition (1) on E to have a fundamental sequence of bounded sets while some variations of the weak barrelledness condition (2) on E have been assumed.

In the present paper we propose another very natural generalization of the concept of (DF) -spaces. We replace the quite strong and demanding condition on E to have a fundamental sequence of bounded sets by a weaker one called a *fundamental bounded resolution* and assume some natural extra property on the weak*-dual of E (which holds for (DF) -spaces), see Definition 1.4.

Let $\{B_n\}_{n \in \mathbb{N}}$ be a fundamental sequence of bounded subsets of a lcs E . For each $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $B_\alpha := B_{n_1}$. Then the family $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the conditions:

- (i) every set B_α is bounded in E ;
- (ii) \mathcal{B} is a resolution in E (i.e., \mathcal{B} covers E and $B_\alpha \subseteq B_\beta$ if $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$);
- (iii) every bounded subset of E is contained in some B_α .

2000 *Mathematics Subject Classification.* Primary 46A03; Secondary 54A25, 54D50.

Key words and phrases. quasi- (DF) -space, (DF) -space, class \mathfrak{G} , fundamental bounded resolution, function space. The third named author supported by GAČR Project 16-34860L and RVO: 67985840.

If a lcs E is covered by a family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded sets satisfying conditions (i) and (ii) [and (iii)] we shall say that E has a [fundamental] bounded resolution, see also [24] for more details. Let us recall that every metrizable lcs E has a fundamental bounded resolution (see, for example, Corollary 2.2 below), while E has a fundamental sequence of bounded sets if and only if E is normable.

The definition of a quasi- (DF) -space involves also the following concept due to Cascales and Orihuela, see [7].

Definition 1.2 (Cascales–Orihuela). A lcs E belongs to the class \mathfrak{G} if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in the weak*-dual $(E', \sigma(E', E))$ of E such that each sequence in any A_α is equicontinuous.

Particularly, every set A_α is relatively $\sigma(E', E)$ -countably compact. The class \mathfrak{G} is indeed large and contains ‘almost all’ important locally convex spaces (including (DF) -spaces and even dual metric spaces). Furthermore the class \mathfrak{G} is stable under taking subspaces, completions, Hausdorff quotients and countable direct sums and products, see [7] or [22].

In [8] the authors introduced and studied a subclass of lcs in the class \mathfrak{G} :

Definition 1.3 (Cascales–Kakol–Saxon). A lcs E is said to have a \mathfrak{G} -base if E admits a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighbourhoods of zero such that $U_\alpha \subseteq U_\beta$ for all $\beta \leq \alpha$.

They proved in [8, Lemma 2] that a quasibarrelled lcs E has a \mathfrak{G} -base if and only if E is in class \mathfrak{G} . However, under $\aleph_1 < \mathfrak{b}$ there is a (DF) -space (hence belongs to the class \mathfrak{G}) which does not admit a \mathfrak{G} -base, see [22] (or Example 4.6 below). Note also that every (LM) -space E has a \mathfrak{G} -base, so E is in the class \mathfrak{G} .

Now we define quasi- (DF) -spaces.

Definition 1.4. A lcs E is called a quasi- (DF) -space if

- (i) E admits a fundamental bounded resolution;
- (ii) E belongs to the class \mathfrak{G} .

Note that (i) and (ii) are independent in the sense that there exist lcs E in the class \mathfrak{G} without a fundamental bounded resolution, and there exist lcs E not being in the class \mathfrak{G} but having a fundamental bounded resolution, see Examples 4.2 and 4.3 below.

The class of quasi- (DF) -spaces is stable under taking subspaces, countable direct sums and countable products (see Theorem 4.1), while infinite products of (DF) -spaces are not of that type.

The organization of the paper goes as follows. Section 2 provides a dual characterization of lcs with fundamental bounded resolutions, see Theorem 2.6. This result nicely applies to show that every regular (LM) -space (for the definition see below) has a fundamental bounded resolution. We provide many concrete examples in order to highlight the differences among some properties or between the concepts which have been introduced, for example concrete examples of lcs having a bounded resolution but not admitting a fundamental bounded resolution will be examined.

Denote by $C_p(X)$ and $C_k(X)$ the space $C(X)$ of continuous real-valued functions on a Tychonoff space X endowed with the pointwise topology and the compact-open topology, respectively. The main result of Section 3 states that $C_p(X)$ has a fundamental bounded resolution if and only if X is countable if and only if the strong dual of $C_p(X)$ has a fundamental bounded resolution, see Theorem 3.2 and Proposition 3.7. This result shows that, although the class \mathfrak{G} of lcs is relatively large, the existence of a fundamental bounded resolution is rather a strong restricted condition for non-metrizable lcs.

Last section gathers some properties of quasi- (DF) -spaces (see Theorem 4.1) and extends a result of Corson in [27]. The previous sections apply to conclude that $C_p(X)$ is a quasi- (DF) -space if and only if $C_p(X)$ has a fundamental bounded resolution if and only if X is countable if and only if the strong dual of $C_p(X)$ is a quasi- (DF) -space. The latter result combined with Theorem 3.2 shows that, for spaces $C_p(X)$, both conditions (i) and (ii) from Definition 1.4 fail for uncountable

X . Recall here that $C_p(X)$ is a (DF)-space only if X is finite, see [37]. On the other hand, if X is a metrizable space, then $C_k(X)$ is a quasi-(DF)-space if and only if X is a Polish σ -compact space, see Proposition 4.8. This shows also an essential difference between quasi-(DF)-spaces and (DF)-spaces $C_k(X)$.

Recall that the strong dual $E'_\beta := (E', \beta(E', E))$ of a metrizable lcs E is a (DF)-space. However, this result fails even for strict (LF)-spaces E since E'_β need not be \aleph_0 -quasibarrelled, see [6, Proposition 1]. Since for a non-metrizable strict (LF)-space E the space E'_β does not have a fundamental sequence of bounded sets, the strong dual E'_β is not a (DF)-space but it is a quasi-(DF)-space, see Theorem 4.1.

2. [FUNDAMENTAL] BOUNDED RESOLUTIONS AND \mathfrak{G} -BASES; GENERAL CASE

In this section we obtain dual characterizations of the existence of a [fundamental] bounded resolutions in a lcs E .

Let I be a partially ordered set. A family $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets of a set Ω is called *I-increasing* (*I-decreasing*) if $A_i \subseteq A_j$ ($A_i \supseteq A_j$, respectively) for every $i \leq j$ in I . We say that the family \mathcal{A} *swallows* a family \mathcal{B} of subsets of Ω if for every $B \in \mathcal{B}$ there is an $i \in I$ such that $B \subseteq A_i$. An $\mathbb{N}^{\mathbb{N}}$ -increasing family of subsets of Ω is called a *resolution* in Ω if it covers Ω . A resolution in a topological space X is called *compact* (respectively, *fundamental compact*) if all its elements are compact subsets of X (and it swallows compact subsets of X , respectively).

We start with the following general observation.

Proposition 2.1. *If a lcs E admits an I-decreasing base at zero, then E has an \mathbb{N}^I -increasing bounded resolution swallowing bounded subsets of E . Consequently E'_β has an \mathbb{N}^I -decreasing base at zero.*

Proof. Let $\mathcal{U} := \{U_i : i \in I\}$ be an I -decreasing base at zero in E . For every $\alpha \in \mathbb{N}^I$, set

$$B_\alpha := \bigcap_{i \in I} \alpha(i)U_i,$$

and set $\mathcal{B} := \{B_\alpha : \alpha \in \mathbb{N}^I\}$. Clearly, \mathcal{B} is \mathbb{N}^I -increasing bounded resolution in E . To show that \mathcal{B} swallows the bounded sets of E , fix a bounded subset B of E . For every $i \in I$, choose a natural number $\alpha(i)$ such that $B \subseteq \alpha(i)U_i$ and set $\alpha := (\alpha(i)) \in \mathbb{N}^I$. Clearly, $B \subseteq B_\alpha$. \square

Since a metrizable lcs has an \mathbb{N} -decreasing base, Proposition 2.1 implies

Corollary 2.2. *If E is a metrizable lcs, then E has a fundamental bounded resolution and hence the space E'_β has a \mathfrak{G} -base.*

Example 2.3. For every uncountable cardinal κ , the space \mathbb{R}^κ does not have bounded resolution. Indeed, assuming the converse we obtain that the complete space \mathbb{R}^κ has a fundamental bounded resolution by Valdivia's theorem [22, Theorem 3.5]. But then the closures of the sets of this latter family compose a fundamental compact resolution. Now Tkachuk's theorem [22, Theorem 9.14] implies that κ is countable, a contradiction.

Next proposition gathers the most important stability properties of spaces with a fundamental bounded resolution.

Proposition 2.4. *The class of locally convex spaces with a fundamental bounded resolution is closed under taking (i) subspaces, (ii) countable direct sums, and (iii) countable products.*

Proof. (i) is clear. To prove (ii) and (iii) we shall use the following encoding operation of elements of $\mathbb{N}^{\mathbb{N}}$. We encode each $\alpha \in \mathbb{N}^{\mathbb{N}}$ into a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ as follows. Consider an arbitrary decomposition of \mathbb{N} onto a disjoint family $\{N_i\}_{i \in \mathbb{N}}$ of infinite sets, where $N_i = \{n_{k,i}\}_{k \in \mathbb{N}}$

for $i \in \mathbb{N}$. Now for $\alpha = (\alpha(n))_{n \in \mathbb{N}}$ and $i \in \mathbb{N}$, we set $\alpha_i = (\alpha_i(k))_{k \in \mathbb{N}}$, where $\alpha_i(k) := \alpha(n_{k,i})$ for every $k \in \mathbb{N}$. Conversely, for every sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$, we define $\alpha = (\alpha(n))_{n \in \mathbb{N}}$ setting $\alpha(n) := \alpha_i(k)$ if $n = n_{k,i}$.

(ii) Let $E = \prod_{i \in \mathbb{N}} E_i$, where every E_i has a fundamental bounded resolution $\{B_\alpha^i : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, we define $B_\alpha := \prod_{i \in \mathbb{N}} B_{\alpha_i}^i$ and set $\mathcal{B} := \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since a subset of E is bounded if and only if its projection onto E_i is bounded in E_i for every $i \in \mathbb{N}$, it is easy to see that \mathcal{B} is a fundamental bounded resolution in E .

(iii) Let $E = \bigoplus_{i \in \mathbb{N}} E_i$, where every E_i has a fundamental bounded resolution $\{B_\alpha^i : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $\alpha^* := (\alpha(k+1))_{k \in \mathbb{N}}$ and $B_\alpha := \prod_{i=1}^{\alpha(1)} B_{\alpha_i^*}^i$. Since every bounded subset of E is contained and bounded in $\bigoplus_{i=1}^m E_i$ for some $m \in \mathbb{N}$ (see Proposition 24 of Chapter 5 of [32]), we obtain that the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution in E . \square

Example 2.5. Let E be an infinite-dimensional Banach space and let B be the closed unit ball of E . Then its dimension is uncountable. Choose a Hamel basis $H = \{x_i^* : i \in I\}$ from its topological dual E' . Then the map $x \rightarrow (x_i^*(x))$ from E_w into \mathbb{R}^H is an embedding with dense image. Since E is a Banach space, the sequence $\{nB\}_{n \in \mathbb{N}}$ is a fundamental bounded sequence in E_w . However, since H is uncountable, the completion \mathbb{R}^H of E_w does not have even a bounded resolution by Example 2.3. Hence, the completion of a lcs with a fundamental bounded resolution in general does not have a bounded resolution.

For a subset A of a lcs E , we denote by A° the polar of A in E' . Below we give a dual characterization of lcs with a fundamental bounded resolution. Recall that a lcs E is a *quasi-(LB)-space* [39] if E admits a resolution consisting of Banach discs of E .

Theorem 2.6. *For a lcs E the following assertions are equivalent:*

- (i) E has a fundamental bounded resolution;
- (ii) the strong dual E'_β of E has a \mathfrak{G} -base;
- (iii) the weak* bidual $(E'', \sigma(E'', E'))$ is a quasi-(LB)-space.

If in addition the space E is locally complete, then (i)-(iii) are equivalent to

- (iv) E has a bounded resolution;
- (v) E is a quasi-(LB)-space.

Proof. (i) \Rightarrow (ii) Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental bounded resolution. Then, as easily seen, the family $\{A_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a \mathfrak{G} -base of neighborhoods of the strong topology on E' .

(ii) \Rightarrow (i) If $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a \mathfrak{G} -base of neighborhoods of $\beta(E', E)$, the polars $\{V_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of the sets V_α in the bidual E'' of E compose a compact resolution on E'' for the weak* topology. Setting $B_\alpha := V_\alpha^\circ \cap E$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, we can see that the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution for $(E, \sigma(E, E'))$, hence for E . If Q is a closed absolutely convex bounded subset of E , the polar Q° of Q in E' is a neighborhood of the origin in $\beta(E', E)$ and consequently there exists $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $V_\beta \subseteq Q^\circ$. This implies that $Q^{\circ\circ} \subseteq V_\beta^\circ$, the polars being taken in E'' . Hence

$$Q = \overline{Q}^{\sigma(E, E')} = Q^{\circ\circ} \cap E \subseteq B_\beta,$$

which means that the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows the bounded sets of E .

(i) \Rightarrow (iii) If $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution of E , it is clear that

$$E'' = \bigcup \{A_\alpha^{\circ\circ} : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$$

Since each $A_\alpha^{\circ\circ}$ is absolutely convex and weak* compact, it is a Banach disc. So $(E'', \sigma(E'', E'))$ has a resolution consisting of weak* compact Banach discs, which means that the weak* bidual $(E'', \sigma(E'', E'))$ is a quasi-(LB)-space.

(iii) \Rightarrow (i) If $(E'', \sigma(E'', E'))$ is a quasi-(LB)-space, then, by [22, Theorem 3.5], there is a resolution $\{D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for $(E'', \sigma(E'', E'))$ consisting of Banach discs that swallows the Banach discs of $(E'', \sigma(E'', E'))$. If Q is a bounded subset of E , then $Q^{\circ\circ}$ is a weak* compact Banach disc in $(E'', \sigma(E'', E'))$. Hence, there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $Q^{\circ\circ} \subseteq D_\gamma$, so that $Q \subseteq D_\gamma \cap E$. Setting $B_\alpha := D_\alpha \cap E$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for E .

Assume that E is locally complete. Clearly, (i) implies (iv). Let us show that (iv) implies (v). Since E is locally complete, the absolutely convex closed envelope of any bounded set of E is a Banach disc by Proposition 5.1.6 of [31]. Thus the space E is a quasi-(LB)-space.

(v) \Rightarrow (i) By Valdivia's theorem [22, Theorem 3.5], there exists another quasi-(LB)-representation $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E swallowing all Banach discs of E . Since each bounded set of E is contained in a Banach disc and each Banach disc is bounded, we obtain that \mathcal{B} is a bounded resolution which swallows all bounded sets of E . \square

Below we apply Theorem 2.6 to function spaces $C_k(X)$ and $C_p(X)$.

Corollary 2.7. *Let X be such that $C_k(X)$ is locally complete (for instance, X is a $k_{\mathbb{R}}$ -space). Then the following assertions are equivalent:*

- (i) $C_k(X)$ has a bounded resolution;
- (ii) $C_k(X)$ has a fundamental bounded resolution.

If in addition X is metrizable, then (i)-(iv) are equivalent to

- (iii) $C_p(X)$ has a bounded resolution;
- (iv) X is σ -compact.

Proof. The equivalences (i)-(ii) follow from Theorem 2.6. The equivalence (iii) \Leftrightarrow (iv) is Corollary 9.2 of [22], and the implication (i) \Rightarrow (iii) is trivial. The implication (iv) \Rightarrow (i) follows from Corollary 2.10 of [17] which states that $C_k(X)$ has even a fundamental compact resolution. \square

Observe that if $C_k(X)$ has a fundamental bounded resolution, $C_k(X)$ need not be metrizable. For instance, $C_k(Q)$ has a fundamental bounded resolution by condition (vi) of the previous corollary, but $C_k(Q)$ is not metrizable.

Corollary 2.8. *Let X be such that $C_k(X)$ is locally complete. If X has an increasing sequence of functionally bounded subsets which swallows the compact sets of X , then the strong dual of $C_k(X)$ has a \mathfrak{G} -base.*

Proof. By Theorem 3.1(ii) of [10], there is a metrizable locally convex topology \mathcal{T} on $C(X)$ stronger than the compact-open topology, and hence $(C(X), \mathcal{T})$ has a fundamental bounded resolution by Corollary 2.2. So $C_k(X)$ has a bounded resolution and Corollary 2.7 applies. \square

Recall that a lcs E is called *quasibarrelled* if every closed absolutely convex bornivorous subset of E is a neighbourhood of zero, see [31] or [5], [21]. Trivially, every metrizable lcs, as well as, any (LM)-space is quasibarrelled.

We supplement Theorem 2.6 with the following fact. Recall that E is *dual locally complete* if $(E', \sigma(E', E))$ is locally complete, see [36]. Note that a lcs E is barrelled if and only if E is a quasibarrelled dual locally complete space, [31].

Proposition 2.9. *The following statements hold true.*

- (i) *Let E be dual locally complete. If E has a \mathfrak{G} -base, then E'_β has a fundamental bounded resolution.*
- (ii) *Let E be a quasibarrelled space. Then the strong dual E'_β of E has a fundamental bounded resolution if and only if E has a \mathfrak{G} -base.*

Proof. (i) Let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a \mathfrak{G} -base in E . Then the polar sets $W_\alpha := U_\alpha^\circ$ are weakly*-compact and absolutely convex, hence Banach discs, so $(E', \sigma(E', E))$ is a quasi- (LB) -space. Again Valdivia's theorem [22, Theorem 3.5] applies to get that $(E', \sigma(E', E))$ has a fundamental resolution $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of Banach discs. As $(E', \sigma(E', E))$ is locally complete, every $\sigma(E', E)$ -bounded set B is included in a Banach disc by Proposition 5.1.6 of [31]. Since Banach discs also are $\beta(E', E)$ -bounded, the family \mathcal{B} is a fundamental bounded resolution.

(ii) Assume that E'_β has a fundamental bounded resolution. Then the strong bidual space E''_β of E has a \mathfrak{G} -base by Theorem 2.6. Since E is quasibarrelled, E is a subspace of E''_β by Theorem 15.2.3 of [28]. Therefore E has a \mathfrak{G} -base. Conversely, assume that E has a \mathfrak{G} -base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then the polar sets $W_\alpha := U_\alpha^\circ$ are weakly*-compact and absolutely convex, and hence W_α are $\beta(E', E)$ -bounded by Theorem 11.11.5 of [28]. To show that $\{W_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution in E' , fix a strongly bounded subset B of E' . Since E is quasibarrelled, W_α is equicontinuous by Theorem 11.11.4 of [28]. So there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $B \subseteq W_\alpha$. \square

A subset A of a Tychonoff space X is *b-bounding* if for every bounded subset B of $C_k(X)$ the number $\sup\{|f(x)| : x \in A, f \in B\}$ is finite. The space X is called a *W-space* if every *b-bounding* subset of X is relatively compact.

Corollary 2.10. *Let X be a W-space (for example, X is realcompact). Then the strong dual E of $C_k(X)$ has a fundamental bounded resolution if and only if X has a fundamental compact resolution.*

Proof. Note that $C_k(X)$ is quasibarrelled by Theorem 10.1.21 of [31]. Therefore, by Proposition 2.9, E has a fundamental bounded resolution if and only if the space $C_k(X)$ has a \mathfrak{G} -base. But $C_k(X)$ has a \mathfrak{G} -base if and only if X has a fundamental compact resolution by [12]. \square

By an (LM) -space $E := (E, \tau)$ we mean a lcs which is the countably inductive limit of an increasing sequence (E_n, τ_n) of metrizable lcs such that $E = \bigcup_n E_n$ and $\tau_n|_{E_n} \leq \tau_n$ for each $n \in \mathbb{N}$. The inductive limit topology τ of E is the finest locally convex topology on E such that $\tau|_{E_n} \leq \tau_n$ for each $n \in \mathbb{N}$. If each step (E_n, τ_n) is a Fréchet lcs, i.e. a metrizable and complete lcs, we call E an (LF) -space. Moreover, if additionally $\tau_{n+1}|_{E_n} = \tau_n$ for each $n \in \mathbb{N}$ the inductive limit space E is called *strict*. The latter case implies that every bounded set in E is contained and bounded in some E_n . Recall that (LM) -spaces enjoying this property are called *regular*. We refer the reader to [31, Definition 8.5.11] or to [21] for details.

Next Proposition 2.11 provides fundamental bounded resolutions for regular (LM) -spaces.

Proposition 2.11. *Let E be an (LM) -space. Then E has a bounded resolution. If in addition E is regular, then E has a fundamental bounded resolution, and consequently, E'_β has a \mathfrak{G} -base.*

Proof. Let (E_i, τ_i) be an increasing sequence of metrizable lcs generating the inductive limit space $E = (E, \tau)$, i. e., $\tau_{i+1}|_{E_i} \leq \tau_i$ and $\tau|_{E_i} \leq \tau_i$ for any $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $\{U_k^i\}_{k \in \mathbb{N}}$ be a decreasing base of neighbourhoods of zero for E_i . For every $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $W_\alpha^i := \bigcap_{k \in \mathbb{N}} \alpha(k)U_k^i$. Any W_α^i is bounded in τ_i , consequently in τ too. It is easy to see that the family $\{W_\alpha^i : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution in E_i . Now, for every $\alpha = (\alpha(k)) \in \mathbb{N}^{\mathbb{N}}$, set $\alpha^* := (\alpha(k+1))$ and $B_\alpha := W_{\alpha^*}^{\alpha(1)}$. Clearly, the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a desired bounded resolution in E .

Assume that E is regular. Then every τ -bounded set is contained in some E_i and is τ_i -bounded. Therefore the bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is fundamental. Finally, the space E'_β has a \mathfrak{G} -base by Theorem 2.6. \square

Corollary 2.12. *Let E be a locally complete lcs which is an image of an infinite-dimensional metrizable topological vector space under a continuous linear map. Then every precompact set in E'_β is metrizable.*

Proof. Let T be a continuous linear map from a metrizable tvs H onto E . It is well-known that H has a bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ (cf. the proof of Proposition 2.1). Then $\{T(B_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution on E . Now Theorem 2.6 implies that E'_β has a \mathfrak{G} -base. Therefore every precompact set in E'_β is metrizable by Cascales–Orihuela’s theorem, see [22]. \square

Remark 2.13. Analysing the proof of Proposition 1 in [11] one can prove the following result: A lcs E has a bounded resolution if and only if E_w is K -analytic-framed in \mathbb{R}^κ for some cardinal κ , that is there exists a K -analytic space H such that $E \subseteq H \subseteq \mathbb{R}^\kappa$.

Remark 2.14. Christensen’s theorem [22, Theorem 6.1] states that a metrizable space X has a fundamental compact resolution if and only if X is Polish. Therefore every *separable* infinite-dimensional Banach space E has a fundamental compact resolution, and this resolution is not a fundamental bounded resolution (otherwise, the closed unit ball B of E would be compact). On the other hand, every *nonseparable* infinite-dimensional metrizable space E has a fundamental *bounded* resolution by Corollary 2.2, but E does not have a fundamental *compact* resolution since the space E is not Polish.

3. MORE ABOUT [FUNDAMENTAL] BOUNDED RESOLUTIONS FOR SPACES $C_p(X)$ AND $C_k(X)$ AND THEIR DUALS

Tkachuk proved (see [22, Theorem 9.14]) that the space $C_p(X)$ has a fundamental compact resolution if and only if X is countable and discrete. Hence, if for an infinite compact space K , the space $C_p(K)$ is K -analytic, then $C_p(K)$ has a compact resolution but it does not have a fundamental compact resolution. Also, if we consider the case $C([0, 1])$, then $C_w([0, 1])$ does not have a fundamental compact resolution, see [26, Corollary 1.10]. Below we prove an analogous result for $C_p(X)$ having a fundamental bounded resolution.

We need the following notion. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, set

$$I_k(\alpha) := \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta(1) = \alpha(1), \dots, \beta(k) = \alpha(k)\}.$$

Definition 3.1. A family $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ of closed subsets of X is called *framing* if

- (1) for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the family $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of X , and
- (2) for every $n \in \mathbb{N}$, $U_{\beta,n} \subseteq U_{\alpha,n}$ whenever $\alpha \leq \beta$.

Recall that a family \mathcal{N} of subsets of a topological space X is called a *cs*-network at a point* $x \in X$ if for each sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; \mathcal{N} is a *cs*-network* in X if \mathcal{N} is a *cs*-network* at each point $x \in X$.

Now we are ready to prove the main result of this section.

Theorem 3.2. *The space $C_p(X)$ admits a fundamental bounded resolution if and only if X is countable (so exactly when $C_p(X)$ is metrizable).*

Proof. If X is countable, the space $C_p(X)$ has a fundamental bounded resolution by Corollary 2.2. Conversely, assume that $C_p(X)$ has a fundamental bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. We prove that X is countable in five steps. First we note the following simple observation.

Step 1. *A subset Q of $C_p(X)$ is bounded if and only if there exists an increasing covering $\{V_n : n \in \mathbb{N}\}$ of X consisting of closed sets such that*

$$\sup_{f \in Q} |f(x)| \leq n \text{ for all } x \in V_n.$$

Indeed, assume that Q is a bounded subset of $C_p(X)$. For every $n \in \mathbb{N}$, set

$$V_n = \{x \in X : \sup_{f \in Q} |f(x)| \leq n\}.$$

Clearly, all V_n are closed, $V_n \subseteq V_{n+1}$ for each $n \in \mathbb{N}$, and $\sup_{f \in Q} |f(x)| \leq n$ for all $x \in V_n$. If $y \in X$ there is $m \in \mathbb{N}$ with $\sup_{f \in Q} |f(y)| \leq m$, so that $y \in V_m$. This shows that $\bigcup_{n \in \mathbb{N}} V_n = X$.

The converse assertion is clear.

Step 2. There exists a framing family $\{U_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ in X enjoying the property that if $\{V_n : n \in \mathbb{N}\}$ is an increasing covering of X consisting of closed sets there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$.

Indeed, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$, set

$$U_{\alpha,n} = \{x \in X : \sup_{f \in B_{\alpha}} |f(x)| \leq n\}.$$

Then, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the family $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing closed covering of X such that $U_{\beta,n} \subseteq U_{\alpha,n}$ whenever $\alpha \leq \beta$, and in addition $\sup_{f \in B_{\alpha}} |f(x)| \leq n$ for every $x \in U_{\alpha,n}$ and $n \in \mathbb{N}$. Therefore the family $\mathcal{U} := \{U_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ is framing.

We claim that \mathcal{U} satisfies the stated property. Indeed, fix an increasing covering $\{V_n : n \in \mathbb{N}\}$ of X consisting of closed sets. Set

$$P := \{f \in C(X) : \sup_{x \in V_n} |f(x)| \leq n \ \forall n \in \mathbb{N}\}.$$

Then P is a bounded subset of $C_p(X)$ by Step 1. Since $\{B_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_p(X)$, there exists $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $P \subseteq B_{\delta}$. To prove the claim we show that $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$. Take arbitrarily $x \in U_{\delta,n}$. Then

$$\sup_{f \in P} |f(x)| \leq \sup_{f \in B_{\delta}} |f(x)| \leq n.$$

Now if $x \notin \overline{V_n} = V_n$, there is $h \in C(X)$ with $0 \leq h \leq n+1$ such that $h(x) = n+1$ and $h(y) = 0$ for every $y \in V_n$. By construction of h and since $\{V_n\}_n$ is increasing, we have $|h(y)| \leq n$ for every $n \in \mathbb{N}$ and each $y \in V_n$. Therefore $h \in P \subseteq B_{\delta}$. So we have at the same time that $x \in U_{\delta,n}$ and $|h(x)| = n+1$ with $h \in B_{\delta}$, a contradiction. Thus $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$.

Step 3. For every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $M_n(\alpha) := \bigcup_{\beta \in I_n(\alpha)} B_{\beta}$ and

$$K_n(\alpha) := \bigcap_{\beta \in I_n(\alpha)} U_{\beta,n} = \left\{ x \in X : \sup_{f \in M_n(\alpha)} |f(x)| \leq n \right\}.$$

Then all $K_n(\alpha)$ are closed (but can be empty) and satisfy the following conditions:

- (i) $K_n(\alpha) \subseteq K_{n+1}(\alpha)$ for every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$;
- (ii) $K_n(\alpha) \supseteq K_n(\beta)$ for every $n \in \mathbb{N}$ whenever $\alpha \leq \beta$;
- (iii) $\bigcup_{n \in \mathbb{N}} K_n(\alpha) = X$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$;
- (iv) for every increasing closed covering $\{V_n : n \in \mathbb{N}\}$ of X there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $K_n(\gamma) \subseteq V_n$ for all $n \in \mathbb{N}$.

Moreover, the family $\mathcal{K} := \{K_n(\alpha) : n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is countable.

Indeed, (i) and (ii) are clear. To prove (iii) suppose for a contradiction that there is $x \notin \bigcup_{n \in \mathbb{N}} K_n(\alpha)$ for some $\alpha \in \mathbb{N}^{\mathbb{N}}$. For every $n \in \mathbb{N}$ choose $\beta_n \in I_n(\alpha)$ such that $x \notin U_{\beta_n,n}$. Set $\gamma := \sup\{\beta_n : n \in \mathbb{N}\}$. Then, for every $n \in \mathbb{N}$, $\beta_n \leq \gamma$ and hence $x \notin U_{\gamma,n}$ since $U_{\gamma,n} \subseteq U_{\beta_n,n}$ by the definition of a framing family. Therefore $x \notin \bigcup_{n \in \mathbb{N}} U_{\gamma,n} = X$, a contradiction. Thus (iii) holds. Now we prove (iv). By the condition of the theorem, there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$. Then $K_n(\gamma) \subseteq V_n$ since $K_n(\gamma) \subseteq U_{\gamma,n}$ for all $n \in \mathbb{N}$. Finally, the family \mathcal{K} is countable since, by construction, the set $K_n(\alpha)$ depends only on $\alpha(1), \dots, \alpha(n)$.

Step 4. For every $m, n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$N_{mn}(\alpha) := \left\{ f \in C(X) : |f(x)| \leq \frac{1}{m} \ \forall x \in K_n(\alpha) \right\}$$

(if $K_n(\alpha)$ is empty we set $N_{mn}(\alpha) := \{0\}$). We claim that the family

$$\mathcal{N} := \left\{ N_{mn}(\alpha) : m, n \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$$

is a countable cs^* -network at $0 \in C_p(X)$.

Indeed, the family \mathcal{N} is countable since the family \mathcal{K} is countable. To show that \mathcal{N} is a cs^* -network at $0 \in C_p(X)$, let $S = \{g_n : n \in \mathbb{N}\}$ be a null-sequence in $C_p(X)$ and let U be a standard neighborhood of zero in $C_p(X)$ of the form

$$U = [F, \varepsilon] := \{f \in C(X) : |f(x)| < \varepsilon \ \forall x \in F\},$$

where F is a finite subset of X and $\varepsilon > 0$. Fix arbitrarily an $m \in \mathbb{N}$ such that $m > 1/\varepsilon$. For every $n \in \mathbb{N}$, set

$$T_n := \bigcap_{i \geq n} R_i, \quad \text{where } R_i := \left\{ x \in X : |g_i(x)| \leq \frac{1}{m} \right\}.$$

It is clear that $\{T_n\}_{n \in \mathbb{N}}$ is an increasing sequence of closed subsets of X . Moreover, since $g_n \rightarrow 0$ in $C_p(X)$ we obtain that $\bigcup_{n \in \mathbb{N}} T_n = X$. Therefore, by (iv), there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that

$$(3.1) \quad K_n(\gamma) \subseteq T_n \text{ for every } n \in \mathbb{N}.$$

Now, by (iii), choose an $n \in \mathbb{N}$ such that $F \subseteq K_n(\gamma)$. Then (3.1) implies

$$\{g_i\}_{i \geq n} \subseteq N_{mn}(\gamma) \subseteq U = [F, \varepsilon].$$

Thus \mathcal{N} is a countable cs^* -network at zero.

Step 5. The space X is countable. Indeed, since the space $C_p(X)$ has a countable cs^* -network at zero by Step 4, the space X is countable by [35] (or [16], recall that $C_p(X)$ is b -Baire-like for every Tychonoff space X). \square

Example 3.3. Let X be an uncountable pseudocompact space. Then the space $C_p(X)$ has a bounded resolution but it does not have a fundamental bounded resolution. Indeed, as X is pseudocompact, the sets $B_n = \{f \in C(X) : |f(x)| \leq n \ \forall x \in X\}$ form a bounded resolution (even a bounded sequence) in $C_p(X)$. The second assertion follows from Theorem 3.2.

For Lindelöf P -spaces we obtain even a stronger result.

Proposition 3.4. *Let X be a Lindelöf P -space. Then $C_p(X)$ has a bounded resolution if and only if X is countable and discrete.*

Proof. Assume that $C_p(X)$ has a bounded resolution. Then $C_p(X)$ is sequentially complete (equivalently locally complete) by [14]. As X is a Lindelöf P -space, the space $C_p(X)$ is Fréchet–Urysohn by Theorem II.7.15 of [3]. By Theorem 14.1 of [22], every sequentially complete Fréchet–Urysohn lcs is Baire, so $C_p(X)$ is a Baire space. But any Baire topological vector space with a bounded resolution is metrizable by Proposition 7.1 of [22]. Therefore $C_p(X)$ is metrizable. Thus X is countable. The converse assertion follows from Corollary 2.2 and the trivial fact that every countable P -space is discrete. \square

Recall (see [27]) that a Tychonoff space X is called a *cosmic space* (an \aleph_0 -space) if X is an image of a separable metric space under a continuous (respectively, compact-covering) map.

Theorem 3.5. *Let $E = C_k(C_k(X))$. Then:*

- (i) *If X is metrizable, then E has a \mathfrak{G} -base if and only if X is σ -compact. In this case E is barrelled.*
- (ii) *If X is an \aleph_0 -space, then the strong dual E'_β of E has a \mathfrak{G} -base if and only if X is finite. In particular, if X is infinite, then E is not a regular (LM)-space.*

- (iii) If X is a μ -space and E'_β has a \mathfrak{G} -base, then X has a fundamental compact resolution, so that $C_k(X)$ has a \mathfrak{G} -base. But the converse is not true in general.

Consequently, if X is an infinite metrizable σ -compact space, then E is a barrelled space with a \mathfrak{G} -base whose strong dual E'_β does not have a \mathfrak{G} -base.

Proof. (i) If E has a \mathfrak{G} -base, then $C_k(X)$ has a fundamental compact resolution by Theorem 2 of [12]. Therefore X is σ -compact by Corollary 9.2 of [22]. Conversely, if X is σ -compact, then $C_k(X)$ has a fundamental compact resolution by Corollary 2.10 of [17]. Once again applying Theorem 2 of [12], we obtain that E has a \mathfrak{G} -base.

To prove the last assertion we note that any metrizable σ -compact space is an \aleph_0 -space, and hence $C_k(X)$ is Lindelöf by [27]. So $C_k(X)$ is a μ -space and the space E is barrelled by the Nachbin–Shirota theorem.

(ii) Assume that the strong dual of $C_k(C_k(X))$ has a \mathfrak{G} -base. Then, by Theorem 2.6, the space $C_k(C_k(X))$ and hence also $C_p(C_k(X))$ have a bounded resolution. Since $C_k(X)$ is also cosmic by Proposition 10.3 of [27], Corollary 9.1 of [22] implies that the space $C_k(X)$ is σ -compact. Therefore $C_p(X)$ is also σ -compact. Now Velichko's theorem [22, Theorem 9.12] implies that X is finite. Conversely, if X is finite and $|X| = n$, then $E = C_k(\mathbb{R}^n)$ is a Fréchet space and Theorem 2.6 applies.

The last assertion follows from this result and Proposition 2.11.

(iii) Let $M_c(X)$ be the topological dual of $C_k(X)$. Denote by \mathcal{T}_k and \mathcal{T}_β the compact-open topology induced from E and the strong topology $\beta(M_c(X), C(X))$ on $M_c(X)$, respectively. Clearly, $\mathcal{T}_k \leq \mathcal{T}_\beta$. Set $G := (M_c(X), \mathcal{T}_k)'$ and $F := (M_c(X), \mathcal{T}_\beta)'$. Then $C(X) \subseteq G \subseteq F$, algebraically. Hence every $\sigma(M_c(X), F)$ -bounded subset of $M_c(X)$ is also $\sigma(M_c(X), G)$ -bounded, and therefore

$$(3.2) \quad \beta(F, M_c(X))|_G \leq \beta(G, M_c(X)).$$

Observe that $C_k(X)$ is barrelled by the Nachbin–Shirota theorem, and hence, by Theorem 15.2.3 of [28], the space $C_k(X)$ is a subspace of its strong bidual space $(F, \beta(F, M_c(X)))$. Therefore

$$(3.3) \quad \tau_k = \beta(F, M_c(X))|_{C(X)}.$$

As $(M_c(X), \mathcal{T}_k)$ is a closed subspace of E we obtain $G = (M_c(X), \mathcal{T}_k)' = E'/M_c(X)^\perp$, algebraically. Denote by \mathcal{T}_q the quotient topology of the strong dual E'_β of E on G . We claim that the strong topology $\beta(G, M_c(X))$ on G is coarser than \mathcal{T}_q . Indeed, if j is the canonical inclusion of $(M_c(X), \mathcal{T}_k)$ into E , the adjoint map j^* is strongly continuous, see [28, Theorem 8.11.3]. Then the claim follows from the fact that j^* is raised to the continuous map from the quotient $E'_\beta/M_c(X)^\perp$ to the strong dual of $(M_c(X), \mathcal{T}_k)$. Now the claim and (3.2) and (3.3) imply

$$(3.4) \quad \tau_k = \beta(F, M_c(X))|_{C(X)} \leq \beta(G, M_c(X))|_{C(X)} \leq \mathcal{T}_q|_{C(X)}.$$

Suppose for a contradiction that E'_β has a \mathfrak{G} -base. Then the quotient topology \mathcal{T}_q and hence $\mathcal{T}_q|_{C(X)}$ also have a \mathfrak{G} -base. Therefore, by (3.4), there exists a locally convex topology $\mathcal{T} := \mathcal{T}_q|_{C(X)}$ with a \mathfrak{G} -base on $C(X)$ such that $\tau_k \leq \mathcal{T}$. According to [10, Corollary 2.3] applied to the family $\mathcal{S} = \mathcal{K}(X)$ of all compact subsets of X and $C(X)$, we obtain that X has a functionally bounded resolution swallowing the compact sets of X . Finally, since X is a μ -space, it follows that X admits a fundamental compact resolution.

Observe that for $X = \mathbb{R}$ the space X is even hemicompact, but E'_β does not have a \mathfrak{G} -base by (ii). \square

Following Markov [25], the free lcs $L(X)$ over a Tychonoff space X is a pair consisting of a lcs $L(X)$ and a continuous mapping $i : X \rightarrow L(X)$ such that every continuous mapping f from X to a lcs E gives rise to a unique continuous linear operator $\bar{f} : L(X) \rightarrow E$ with $f = \bar{f} \circ i$. The free lcs $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding. Denote by $L_p(X)$ the free lcs $L(X)$ endowed with the weak topology.

Following [4], a Tychonoff space X is called an *Ascoli space* if every compact subset \mathcal{K} of $C_k(X)$ is equicontinuous (see [15]). By Ascoli's theorem [28], each k -space is Ascoli. The following theorem complements and extends Theorem 3.2 and Corollaries 3.3 and 3.4 of [17].

Theorem 3.6. *For an Ascoli space X the following assertions are equivalent:*

- (i) $L(X)$ has a \mathfrak{G} -base;
- (ii) $C_k(X)$ has a fundamental compact resolution;
- (iii) $C_k(C_k(X))$ has a \mathfrak{G} -base.

In particular, any item above implies that every compact subset of X is metrizable.

Proof. (i) \Rightarrow (ii) Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a \mathfrak{G} -base in $L(X)$. Then the family $\mathcal{U}^\circ = \{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where the polars are taken in the dual space $L(X)'$ of $L(X)$, is a compact resolution of $(L(X)', \sigma(L(X)', L(X)))$. It is well known and easy to see that the dual space $L(X)'$ of $L(X)$ can be identified with the space $C(X)$ under the restriction map (recall that X is a Hamel base for $L(X)$)

$$L(X)' \ni \chi \mapsto \chi|_X \in C(X).$$

It is clear that the weak* topology $\sigma(L(X)', L(X))$ on $L(X)'$ induces the pointwise topology τ_p on $C(X)$. Therefore, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, the set $K_\alpha := \{\chi|_X : \chi \in U_\alpha^\circ\}$ is closed also in the compact-open topology τ_k on $C(X)$. We claim that K_α is a compact subset of $C_k(X)$. For this, by the Ascoli theorem [28, Theorem 5.10.4], it is sufficient to check that K_α is pointwise bounded and equicontinuous. Clearly, K_α is pointwise bounded. To show that K_α is equicontinuous fix an $x \in X$. Then for every $y = x + t \in (x + U_\alpha) \cap X$ (recall that X is a subspace of $L(X)$), we obtain

$$|\chi|_X(y) - \chi|_X(x)| = |\chi(t)| \leq 1, \quad \forall \chi|_X \in K_\alpha.$$

Thus K_α is a compact subset of $C_k(X)$. Consequently, the family $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $C_k(X)$.

Take arbitrarily a compact subset K of $C_k(X)$. Then the polar K° of K in $C_k(C_k(X))$ is a neighborhood of zero in $C_k(C_k(X))$. Since X is Ascoli, the space $L(X)$ is a subspace of $C_k(C_k(X))$ by Theorem 1.2 of [15]. So there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $U_\alpha \subset K^\circ \cap L(X)$. Hence $K \subseteq K^{\circ\circ} \subseteq U_\alpha^\circ$. Thus \mathcal{K} swallows the compact sets of $C_k(X)$.

(ii) \Rightarrow (iii) follows from Theorem 2 of [12], and (iii) implies (i) since $L(X)$ is a subspace of $C_k(C_k(X))$ by Theorem 1.2 of [15].

The last assertion follows from the fact that X is a subspace of $L(X)$ and Cascales–Orihuela's theorem [7, Theorem 11] (which states that every compact subset of a lcs with a \mathfrak{G} -base is metrizable). \square

The previous theorem may suggest the following problem: Characterize in terms of X those spaces $C_k(X)$ with a fundamental bounded resolution.

We propose also the following

Proposition 3.7. *The following assertions are equivalent:*

- (i) The strong dual space $L(X)_\beta$ of $C_p(X)$ has a bounded resolution.
- (ii) $L(X)_\beta$ has a fundamental bounded resolution.
- (iii) X is countable.

Proof. (i) \Rightarrow (iii) Assume that $L(X)_\beta$ has a bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since $C_p(X)$ is quasibarrelled, its strong dual $L(X)_\beta$ is feral, see p. 392 of [13] (recall that following [23], a lcs E is called *feral* if every bounded set of E is finite-dimensional). If X has an uncountable number of points, some set B_α of the resolution would contain infinitely many points of X by [22, Proposition 3.7]. Since X is a Hamel basis for $L(X)$, the set B_α would be infinite-dimensional, a contradiction. Thus X must be countable.

(iii) \Rightarrow (ii) If X is countable, $C_p(X)$ is metrizable. Thus E has a fundamental bounded resolution by Proposition 2.9. Finally, the implication (ii) \Rightarrow (i) is trivial. \square

The following item characterizes those μ -spaces X for which the weak* dual $L_p(X)$ of $C_p(X)$ has a fundamental bounded resolution.

Proposition 3.8. *Let X be a μ -space. Then X has a fundamental compact resolution if and only if $L_p(X)$ has a fundamental bounded resolution.*

Proof. Denote by $\delta : X \rightarrow L_p(X)$ the canonical embedding map and let E be the topological dual of $C_k(X)$. First we observe that if X is a μ -space, then $C_k(X)$ is the strong dual of $L_p(X)$. Indeed, since X is a μ -space, the barrelledness of $C_k(X)$ yields $\tau_k = \beta(C(X), E)$. Since clearly $\tau_k \leq \beta(C(X), L(X)) \leq \beta(C(X), E)$, it follows that $\tau_k = \beta(C(X), L(X))$.

We claim that if A is a bounded subset of $L_p(X)$, there is a compact set K in X and $n \in \mathbb{N}$ such that $A \subseteq n \cdot \overline{\text{abx}(\delta(K))}^{L_p(X)}$. Indeed, by the previous observation, if A is a bounded set in $L_p(X)$ there are $n \in \mathbb{N}$ and a compact subset K of X such that

$$\{f \in C(X) : \sup_{x \in K} |f(x)| \leq n^{-1}\} \subseteq A^\circ.$$

So, if u_f denotes the (unique) continuous linear extension of f to $L_p(X)$, we have

$$\delta(K)^\circ = \{f \in C(X) : \sup_{x \in K} |\langle \delta_x, u_f \rangle| \leq 1\} \subseteq nA^\circ.$$

Hence $A \subseteq A^{\circ\circ} \subseteq n \delta(K)^{\circ\circ}$, where the bipolar is taken in E . Thus $A \subseteq n \cdot \overline{\text{abx}(\delta(K))}^{L_p(X)}$.

Assume that X has a fundamental compact resolution $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For every $\alpha = (\alpha(i)) \in \mathbb{N}^{\mathbb{N}}$, set $\alpha^* := (\alpha(i+1))$ and

$$B_\alpha := \alpha(1) \cdot \overline{\text{abx}(\delta(K_{\alpha^*}))}^{L_p(X)}.$$

Clearly the family $\mathcal{B} := \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consists of bounded sets in $L_p(X)$ and satisfies that $B_\alpha \subseteq B_\beta$ whenever $\alpha \leq \beta$. Moreover, according to the claim and the fact that the family \mathcal{K} is fundamental, we obtain that \mathcal{B} is a fundamental bounded resolution for $L_p(X)$.

Conversely, let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental bounded resolution for $L_p(X)$. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $M_\alpha := A_\alpha \cap \delta(X)$. Then $\{M_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution for $\delta(X)$ consisting of functionally bounded sets that swallows the functionally bounded subsets of $\delta(X)$. Since X is a μ -space, then the family $\{\overline{\delta^{-1}(M_\alpha)}^X : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental compact resolution for X . \square

The condition that X is a μ -space cannot be removed from Proposition 3.8, as the following example shows.

Example 3.9. Let X be the ordinal space $[0, \omega_1)$, where ω_1 is the first uncountable ordinal, and set $Y := [0, \omega_1]$. Since X is pseudocompact, the restriction map $Tf = f|_X$ maps continuously $C_p(Y)$ onto $C_p(X)$. Consequently, $L_p(X)$ is topologically isomorphic to a linear subspace of $L_p(Y)$. Since Y is a compact set, according to Proposition 3.8 the space $L_p(Y)$ has a (countable) fundamental bounded resolution, which implies that $L_p(X)$ also has a fundamental bounded resolution. However, under $MA + \neg CH$ the space $X = [0, \omega_1)$ even does not have a compact resolution [38, Theorem 3.6]. Observe that X is not a μ -space.

Next example shows that there is no natural relationship between the existence of fundamental bounded resolutions for different natural topologies.

Example 3.10. If X is an uncountable Polish space, then the strong dual of $C_p(X)$ does not have a fundamental bounded resolution by Proposition 3.7, but the weak* dual of $C_p(X)$ has a fundamental bounded resolution by Proposition 3.8 and Christensen's theorem (see, [22, Theorem 6.1]). On the other hand, if X is a countable but non-Polish metrizable space (for example, $X = \mathbb{Q}$), then the

strong dual of $C_p(X)$ has a fundamental bounded resolution by Proposition 3.7, however the weak* dual of $C_p(X)$ does not have a fundamental bounded resolution by Proposition 3.8 and Christensen's theorem.

4. QUASI-(DF)-SPACES

In Introduction we formally defined the class of quasi-(DF)-spaces. The previous sections apply to gather a few fundamental properties of quasi-(DF)-spaces. We refer the readers to corresponding facts dealing with (DF)-spaces, see [21] and [31]. Note however that quasi-(DF)-spaces are stable by taking countable products although this property fails for (DF)-spaces.

Theorem 4.1. *The following statements hold true.*

- (i) *Every regular (LM)-space E is a quasi-(DF)-space. In particular, every infinite-dimensional metrizable non-normable lcs E is a quasi-(DF)-space not being a (DF)-space.*
- (ii) *The strong dual E'_β of a regular (LM)-space is a quasi-(DF)-space.*
- (iii) *Countable direct sums of quasi-(DF)-spaces are quasi-(DF)-spaces.*
- (iv) *A subspace of a quasi-(DF)-space is a quasi-(DF)-space.*
- (v) *A countable product E of quasi-(DF)-spaces is a quasi-(DF)-space.*
- (vi) *Every precompact set of a quasi-(DF)-space is metrizable.*

Proof. (i) Every (LM)-space belongs to the class \mathfrak{G} , see [22, Section 11.1]. Being regular, E has a fundamental bounded resolution by Proposition 2.11. Thus E is a quasi-(DF)-space. In particular, if E is an infinite-dimensional metrizable non-normable lcs, then E is a quasi-(DF)-space which is not a (DF)-space.

(ii) Since E is regular, the space E'_β has a \mathfrak{G} -base by Proposition 2.11. Therefore E'_β is in the class \mathfrak{G} (recall that every lcs E with a \mathfrak{G} base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ belongs to the class \mathfrak{G} since the family of polars $\{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a \mathfrak{G} -representation of E). As E has a \mathfrak{G} -base and is quasibarrelled (E is even bornological, see [21, Corollary 13.1.5]), the space E'_β has a fundamental bounded resolution by Proposition 2.9. Therefore E'_β is a quasi-(DF)-space.

(iii)-(v) follow from [22, Proposition 11.1] and Proposition 2.4, and (vi) follows from [22, Theorem 11.1]. \square

Next two examples show the independence of conditions (i) and (ii) appearing in Definition 1.4 of quasi-(DF)-spaces.

Example 4.2. There exist lcs E not being in class \mathfrak{G} but having a fundamental bounded resolution. Indeed, if E is an infinite-dimensional Banach space, then E_w is not in class \mathfrak{G} (see [22]) although E_w has a fundamental sequence of bounded sets: Assume E_w is in class \mathfrak{G} . Then, as E_w is dense in \mathbb{R}^X for some X , the Baire space \mathbb{R}^X belongs also to the class \mathfrak{G} . Now the main theorem of [24] applies to deduce that X is countable, a contradiction (since then E_w would be metrizable implying the finite-dimensionality of E).

Example 4.3. Let X be a non σ -compact Čech-complete Lindelöf space (for example, $X = \mathbb{N}^{\mathbb{N}}$). Then $C_k(X)$ has a \mathfrak{G} -base (hence is in the class \mathfrak{G}) and is barrelled but it does not have even a bounded resolution. Consequently, $C_k(X)$ is not a quasi-(DF)-space and the strong dual of $C_k(X)$ does not have a \mathfrak{G} -base.

Proof. Since X has a fundamental compact resolution by (see Fact 1 in the proof of Proposition 4.7 in [19]), the space $C_k(X)$ has a \mathfrak{G} -base by [12]. As X is a μ -space, $C_k(X)$ is barrelled. On the other hand, assume that $C_k(X)$ has a bounded resolution. Then $C_p(X)$ has a bounded resolution too. Since X is Čech-complete and Lindelöf, there exists (well known fact) a perfect map T from X onto a Polish space Y . As X is not σ -compact, then Y is also not σ -compact. Since T is onto, the adjoint map $T^* : C_p(Y) \rightarrow C_p(X)$, $T^*(f) = f \circ T$, of T is an embedding. Therefore $C_p(Y)$ has

a bounded resolution. But this is impossible, since then Y would be σ -compact by Corollary 9.2 of [22]. The last assertion follows from Theorem 2.6. \square

In [27] it is proved that if E is a Banach space whose strong dual is separable, then E_w is an \aleph_0 -space. In [18, Corollary 5.6] it was shown that a Banach space which does not contain an isomorphic copy of ℓ_1 has separable dual if and only if E_w is an \aleph_0 -space. Next theorem extends this result to quasi- (DF) -spaces.

Theorem 4.4. *Let E be a quasi- (DF) -space.*

- (i) *If the strong dual E'_β is separable, then E_w is cosmic.*
- (ii) *If the strong dual E'_β is separable and barrelled, then E_w is an \aleph_0 -space.*
- (iii) *If E is a strict (LF) -space such that E'_β is separable, then E_w is an \aleph_0 -space.*

Proof. (i) By Theorem 2.6, the space E'_β has a \mathfrak{G} -base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Hence its polar $\{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ forms a compact resolution in $E''_w := (E'', \sigma(E'', E'))$. Since E'_β is separable, $\sigma(E'', E')$ admits a weaker metrizable topology, and hence the space E''_w is analytic by [7, Theorem 15] (i.e. E''_w is a continuous image on $\mathbb{N}^{\mathbb{N}}$). Therefore E''_w is a cosmic space. As E_w is a subspace of E''_w , the space E_w is cosmic as well.

(ii) As in (i), the space E''_w has a compact resolution $\{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. We show that E''_w has a fundamental compact resolution. Indeed, Theorem 2.6 and the fact that E''_w is locally complete (since E'_β is barrelled) imply that E''_w has a fundamental bounded resolution. Now, by the barrelledness of E'_β , the space E'_β has a \mathfrak{G} -base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Therefore the family $\mathcal{U}^\circ := \{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution for E''_w consisting of compact subsets. To check that \mathcal{U}° swallows the compact sets, let K be a compact subset of E''_w . As E'_β is barrelled, K is equicontinuous. So there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $K \subseteq U_\alpha^\circ$, and hence \mathcal{U}° swallows the compact sets. On the other hand, E''_w is submetrizable since E'_β is separable. Now Theorem 3.6 of [9] yields that E''_w is an \aleph_0 -space, so E_w is an \aleph_0 -space, too.

(iii) Any strict (LF) -space E , being regular, is a quasi- (DF) -space by (i) of Theorem 4.1. Note also that any (LM) -space is quasibarrelled (even bornological). Thus to apply (ii) it is sufficient to show that E'_β is barrelled. Let $\{E_n\}_{n \in \mathbb{N}}$ be a defining sequence of Fréchet lcs for E . For each $n \in \mathbb{N}$, the strong dual $(E_n)'_\beta$ of E_n is a (DF) -space. Since E is a strict limit, the dual E'_β is linearly homeomorphic with the projective limit of the sequence $\{(E_n)'_\beta\}_{n \in \mathbb{N}}$ of complete (DF) -spaces, see [6, Preliminaries]. Therefore E'_β is also complete. On the other hand, E'_β is continuously mapped onto each $(E_n)'_\beta$, so any $(E_n)'_\beta$ is separable. By [31, Proposition 8.3.45], any E_n is distinguished. Hence applying again [6, Preliminaries (c)], the space E'_β is quasibarrelled. Since any complete quasibarrelled space is barrelled (see [21, Proposition 11.2.4]), the space E'_β is barrelled. \square

Below we provide more concrete examples which clarify the fundamental differences between (DF) -spaces and quasi- (DF) -spaces.

Example 4.5. *The space of distributions $D'(\Omega)$ over an open non-empty subset Ω of \mathbb{R}^n has the following properties:*

- (i) *$D'(\Omega)$ is a quasi- (DF) -space.*
- (ii) *$D'(\Omega)$ is a weakly \aleph_0 -space.*
- (iii) *$D'(\Omega)$ is not a (DF) -space.*
- (iv) *$D'(\Omega)$ is not a weakly Ascoli space.*

Proof. Recall that the space $D'(\Omega)$ is the strong dual of the space $D(\Omega)$ of test functions which is a complete Montel (hence barrelled) strict (LF) -space of a sequence of Montel–Fréchet lcs. Therefore $D'(\Omega)$ is a quasi- (DF) -space by (ii) of Theorem 4.1. Since $D'(\Omega)$ is a Montel quasi- (DF) -space whose strong dual $D(\Omega)$ is separable and barrelled, $D'(\Omega)$ is a weakly \aleph_0 -space by

(ii) of Theorem 4.4. As $D'(\Omega)$ does not have a fundamental bounded sequence (otherwise $D(\Omega)$ would be metrizable), $D'(\Omega)$ is not a (DF)-space. Finally, Theorem 1.6 of [15] states that if E is a barrelled weakly Ascoli space, then every weak*-bounded subset of E' is finite-dimensional. Thus $D'(\Omega)$ is not weakly Ascoli since $D(\Omega)$ has an infinite-dimensional compact sets. \square

Example 4.6. (Under $\aleph_1 < \mathfrak{b}$) The space $C_k(\omega_1)$ is a (DF)-space by Theorem 12.6.4 of [21] which is not barrelled (recall that the ordinal space $w_1 = [0, \omega_1)$ is pseudocompact). However, $C_k(\omega_1)$ does not have a \mathfrak{G} -base by Proposition 16.13 of [22].

As we mentioned above, it is known that the space $C_p(X)$ is a (DF)-space if and only if X is finite. Indeed, although always $C_p(X)$ is quasibarrelled, see [21, Corollary 11.7.3], the space $C_p(X)$ admits a fundamental sequence of bounded sets only if X is finite, see [22]. For quasi-(DF)-spaces $C_p(X)$ the corresponding situation looks even more striking as the following theorem shows.

Theorem 4.7. *For $C_p(X)$ the following assertions are equivalent:*

- (i) $C_p(X)$ is a quasi-(DF)-space.
- (ii) $C_p(X)$ has a fundamental bounded resolution.
- (iii) X is countable.
- (iv) $C_p(X)$ is in the class \mathfrak{G} .
- (v) $C_p(X)$ has a \mathfrak{G} -base.

Proof. (ii) follows from (i) by the definition. (iii) follows from (ii) by Theorem 3.2. The implication (iii) \Rightarrow (iv) is trivial. Since $C_p(X)$ is always quasibarrelled (see again [21, Corollary 11.7.3]), the implication (iv) \Rightarrow (v) follows from [8]. Finally, if $C_p(X)$ has a \mathfrak{G} -base, it is metrizable again by [8]. \square

Theorem 4.7 suggests the following question: *For which Tychonoff space X , the space $C_k(X)$ is a quasi-(DF)-space?* Below we obtain a complete answer to this question for metrizable spaces X .

Proposition 4.8. *Let X be a metrizable space. Then $C_k(X)$ is a quasi-(DF)-space if and only if X is a Polish σ -compact space. In particular, if X is a Polish σ -compact but non-compact space, then $C_k(X)$ is a quasi-(DF)-space which is not a (DF)-space.*

Proof. Assume that $C_k(X)$ is a quasi-(DF)-space. Since $C_k(X)$ has a fundamental bounded resolution, Corollary 2.7 implies that X is a σ -compact space. On the other hand, as $C_k(X)$ is barrelled, $C_k(X)$ belongs to the class \mathfrak{G} if and only if it has a \mathfrak{G} -base, see Lemma 15.2 of [22]. Therefore X has a fundamental compact resolution by [12]. Thus X is Polish by the Christensen theorem [22, Theorem 6.1].

Conversely, if X is a Polish σ -compact space, then $C_k(X)$ has a \mathfrak{G} -base by [12] and has a fundamental bounded resolution by Corollary 2.7. Thus $C_k(X)$ is a quasi-(DF)-space.

If the Polish σ -compact X is not compact, it has a countable non relatively compact subset. Thus $C_k(X)$ is a (DF)-space by Theorem 10.1.22 of [31]. \square

Recall that $C_k(X)$ is a (DF)-space if and only if any countable union of compact subsets of X is a relatively compact set, see [31, Theorem 10.1.22]. It is known that a (DF)-space E is quasibarrelled if and only if E has countable tightness, see [22, Proposition 16.4 and Theorem 12.3]. Therefore any vector subspace of a quasibarrelled (DF)-space has the same property. This may suggest also the following

Question 4.9. *Does there exist a quasi-(DF)-space with countable tightness not being quasibarrelled?*

REFERENCES

1. N. Adasch, B. Ernst, Lokaltopologische Vektorräume, *Collectanea Math.* **25** (1974), 255–274.
2. N. Adasch, B. Ernst, Lokaltopologische Vektorräume II, *Collectanea Math.* **26** (1975), 13–18.
3. A.V. Arhangel'skii, *Topological function spaces*, Math. Appl. **78**, Kluwer Academic Publishers, Dordrecht, 1992.
4. T. Banach, S. Gabrielyan, On the C_k -stable closure of the class of (separable) metrizable spaces, *Monatshefte Math.* **180** (2016), 39–64.
5. V.I. Bogachev, O.G. Smolyanov, *Topological Vector Spaces and Their Applications*, Springer Monographs in Mathematics, 2017.
6. J. Bonet, S. Dierolf, A note on biduals of strict (LF) -spaces, *Results Math.* **13** (1988), 23–32.
7. B. Cascales, J. Orihuela, On compactness in locally convex spaces, *Math. Z.* **195** (1987), 365–381.
8. B. Cascales, J. Kąkol and S. A. Saxon, Metrizable vs. Fréchet-Urysohn property, *Proc. Amer. Math. Soc.* **131** (2003), 3623–3631.
9. B. Cascales, J. Orihuela, V. Tkachuk, Domination by second countable spaces and Lindelöf Σ -property, *Topology Appl.* **158** (2011), 204–214.
10. J.C. Ferrando, S. Gabrielyan, J. Kąkol, Metrizable-like locally convex topologies on $C(X)$, *Topology Appl.* **230** (2017), 105–113.
11. J.C. Ferrando, J. Kąkol, A note on spaces $C_p(X)$ K -analytic-framed in \mathbb{R}^X , *Bull. Austral. Math. Soc.* **78** (2008), 141–146.
12. J.C. Ferrando, J. Kąkol, On precompact sets in spaces $C_c(X)$, *Georgian Math. J.* **20** (2013), 247–254.
13. J.C. Ferrando, J. Kąkol, S. Saxon, The dual of the locally convex space $C_p(X)$, *Funct. Approx. Comment. Math.* **50** (2014), 389–399.
14. J.C. Ferrando, J. Kąkol, S. Saxon, Characterizing P -spaces X in terms of $C_p(X)$, *J. Convex Anal.* **22** (2015), 905–915.
15. S. Gabrielyan, On the Ascoli property for locally convex spaces, *Topology Appl.* **230** (2017), 517–530.
16. S. Gabrielyan, J. Kąkol, Metrization conditions for topological vector spaces with Baire type properties, *Topology Appl.* **173** (2014), 135–141.
17. S. Gabrielyan, J. Kąkol, Free locally convex spaces with a small base, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas RACSAM*, **207** (2017), 136–155.
18. S. Gabrielyan, J. Kąkol, W. Kubiś, W. Marciszewski, Networks for the weak topology of Banach and Fréchet spaces, *J. Math. Anal. Appl.* **432** (2015), 1183–1199.
19. S. Gabrielyan, J. Kąkol, A. Kubzdela, M. Lopez Pellicer, On topological properties of Fréchet locally convex spaces with the weak topology, *Topology Appl.* **192** (2015), 123–137.
20. A. Grothendieck, Sur les espaces (F) et (DF) , *Summa Brasil. Math.* **3** (1954), 57–123.
21. H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981.
22. J. Kąkol, W. Kubiś, M. Lopez-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Developments in Mathematics, Springer, 2011.
23. J. Kąkol, S.A. Saxon, A.R. Todd, Weak barrelledness for $C(X)$ spaces, *J. Math. Anal. Appl.* **297** (2004), 495–505.
24. J. Kąkol, M. Lopez-Pellicer, Compact covering for Baire locally convex spaces, *J. Math. Anal. Appl.* **332** (2007), 965–974.
25. A.A. Markov, On free topological groups, *Dokl. Akad. Nauk SSSR* **31** (1941), 299–301.
26. S. Mercourakis, E. Stamati, A new class of weakly K -analytic Banach spaces, *Comment. Math. Univ. Carolin.* **47** (2006), 291–312.
27. E. Michael, \aleph_0 -spaces, *J. Math. Mech.* **15** (1966), 983–1002.
28. L. Narici, E. Beckenstein, *Topological vector spaces*, Second Edition, CRC Press, New York, 2011.
29. K. Nouredine, Nouvelles classes d'espaces localement convexes, *Publ. Dept. Math. Lyon* **10-3** (1973), 259–277.
30. K. Nouredine, Notes sur les espaces D_b , *Math. Ann.* **219** (1976), 97–103.
31. P. Pérez Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies **131**, Amsterdam, 1987.
32. A.P. Robertson, W.J. Robertson, *Topological Vector Spaces*, second edition, Cambridge University Press, 1964.
33. W. Ruess, *Halbnorm-Dualität und induktive Limestopologien in der Theorie lokalkonvexer Räume*, *Habil. schr. Univ. Bonn.* 1976.
34. W. Ruess, On the locally convex structure of strict topologies, *Math. Z.* **153** (1977), 179–192.
35. M. Sakai, Function spaces with a countable cs^* -network at a point, *Topology Appl.* **156** (2008), 117–123.
36. S.A. Saxon, L.M. Sánchez Ruiz, Dual Local Completeness, *Proc. Amer. Math. Soc.*, **125** (1997), 1063–1070.
37. J. Schmets, *Espaces de fonctions continues*, **519**, Springer-Verlag, Berlin, New York, 1976, *Lecture Notes in Mathematics*.
38. V.V. Tkachuk, A space $C_p(X)$ is dominated by irrationals if and only if it is K -analytic, *Acta Math. Hung.* **107** (2005), 253–265.

39. M. Valdivia, Quasi-(LB)-spaces, J. London Math. Soc. **35** (1987), 149–168.

CENTRO DE INVESTIGACIÓN OPERATIVA, EDIFICIO TORRETAMARIT, AVDA DE LA UNIVERSIDAD, ELCHE, SPAIN
E-mail address: `jc.ferrando@umh.es`

DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA, P.O. 653, ISRAEL
E-mail address: `saak@math.bgu.ac.il`

FACULTY OF MATHEMATICS AND INFORMATICS. A. MICKIEWICZ UNIVERSITY, 61-614 POZNAŃ, AND INSTITUTE
OF MATHEMATICS CZECH ACADEMY OF SCIENCES, PRAGUE
E-mail address: `kakol@amu.edu.pl`