

ON TOPOLOGICAL PROPERTIES OF THE GROUP OF THE NULL SEQUENCES VALUED IN AN ABELIAN TOPOLOGICAL GROUP

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ABSTRACT. Following [23], denote by \mathfrak{F}_0 the functor on the category **TAG** of all Hausdorff Abelian topological groups and continuous homomorphisms which passes each $X \in \mathbf{TAG}$ to the group of all X -valued null sequences endowed with the uniform topology. We prove that if $X \in \mathbf{TAG}$ is an (E) -space (respectively, a strictly angelic space or a \check{S} -space), then $\mathfrak{F}_0(X)$ is an (E) -space (respectively, a strictly angelic space or a \check{S} -space). We study respected properties for topological groups in particular from categorical point of view. Using this investigation we show that for a locally compact Abelian (LCA) group X the following are equivalent: 1) X is totally disconnected, 2) $\mathfrak{F}_0(X)$ is a Schwartz group, 3) $\mathfrak{F}_0(X)$ respects compactness, 4) $\mathfrak{F}_0(X)$ has the Schur property. So, if a LCA group X has non-zero connected component, the group $\mathfrak{F}_0(X)$ is a reflexive non-Schwartz group which does not have the Schur property. We prove also that for every compact connected metrizable Abelian group X the group $\mathfrak{F}_0(X)$ is monothetic that generalizes a result by Rolewicz for $X = \mathbb{T}$.

1. INTRODUCTION

Properties and functors in TAG. Denote by **TAG** the category of all Hausdorff Abelian topological groups and continuous homomorphisms. Below we consider some important subcategories, functors and properties in **TAG** which are essentially used in the article.

(A) *Properties in TAG.* We consider the following three types of properties in **TAG**. The properties of the type \mathcal{P}_s are those ones which are preserved under taking closed subgroups (i.e., if $X \in \mathbf{TAG}$ has a property \mathcal{P} and H is a closed subgroup of X , then H also has \mathcal{P}): (pre)compactness, local (pre)compactness, realcompactness, sequential compactness, countable compactness, completeness, sequential completeness, normality, sequentiality, Fréchet-Urycohnness, metrizability etc. [1, 28, 31, 39]. The properties which are preserved under taking quotients form the type \mathcal{P}_q : connectedness and pseudocompactness etc. [1, 31, 39]. We denote by \mathcal{P}_n the properties which are not preserved under taking neither closed subgroups nor quotients: (Pontryagin) reflexivity etc. (see, the survey [12]).

Let \mathbf{A} be a subcategory of **TAG** and let $F : \mathbf{TAG} \rightarrow \mathbf{TAG}$ be a functor. We say that \mathbf{A} is *F-invariant* if $F(X) \in \mathbf{A}$ for every $X \in \mathbf{A}$. We say that the functor F *preserves a property \mathcal{P} on \mathbf{A}* if $F(X) \in \mathbf{TAG} \cap \mathcal{P}$ for each $X \in \mathbf{A} \cap \mathcal{P}$. The next two general problems are natural:

Problem 1.1. *Characterize all groups $X \in \mathbf{A} \cap \mathcal{P}$ such that $F(X) \in \mathbf{TAG} \cap \mathcal{P}$. Does the functor F preserve the property \mathcal{P} on \mathbf{A} ?*

Problem 1.2. *Let $F(X) \in \mathbf{A} \cap \mathcal{P}$ for $X \in \mathbf{A}$. Does also $X \in \mathbf{A} \cap \mathcal{P}$?*

(B) *The Bohr functor.* For an Abelian topological group X we denoted by \widehat{X} the group of all continuous characters on X . The group X is called *maximally almost periodic* (MAP) if \widehat{X} separates the points of X . Denote by **MAPA** the full subcategory of **TAG** consisting of all MAP Abelian groups. The class **LCA** of all locally compact Abelian (LCA for short) groups is the most important full subcategory of **MAPA**. For $(X, \tau) \in \mathbf{MAPA}$ we denote by $\sigma(X, \widehat{X})$ or τ^+ the *weak topology* on X , i.e., the smallest group topology in X for which the elements of \widehat{X} are continuous. The topology τ^+ is called also *the Bohr modification* of τ . Denote by $\mathfrak{B} : \mathbf{MAPA} \rightarrow \mathbf{MAPA}$ the Bohr functor, i.e., $\mathfrak{B}(X, \tau) := X^+$ for every $(X, \tau) \in \mathbf{MAPA}$, where

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$X^+ := (X, \tau^+)$. Note that \mathfrak{B} is finitely multiplicative. It is well-known that the groups X and X^+ have the same set of continuous characters, and $\mathfrak{B}(X) = X$ if and only if X is precompact (see [1, 16]).

It is known that, on the class **LCA** the Bohr functor \mathfrak{B} preserves connectedness [67], covering dimension [44, 67] and realcompactness [14]. On the other hand, X^+ is normal if and only if X is σ -compact [68]. Note also that, if X is discrete and infinite, then X^+ is sequentially complete [28] and non-pseudocompact [18]. The same holds for the general case: if X is a non-compact LCA group, then X^+ is sequentially complete and non-pseudocompact [22].

(C) *Respected properties.* Using the Bohr functor we can define *respected properties* in subcategories of **MAPA**. Following [57], if \mathcal{P} denotes a topological property, then we say that a *MAPA* group X *respects* \mathcal{P} if X and X^+ have the same subspaces with \mathcal{P} . Let \mathbf{A} be a subcategory of **MAPA**.

Problem 1.3. *Characterize $X \in \mathbf{A}$ which respect \mathcal{P} .*

We say that \mathcal{P} is a *respected property* in \mathbf{A} if every $X \in \mathbf{A}$ respects \mathcal{P} .

Problem 1.4. *Which topological properties are respected for a given subcategory \mathbf{A} of **MAPA**?*

Recall that $X \in \mathbf{MAPA}$ is said to have: 1) the *Glicksberg property* or *X respects compactness* if the compact subsets of X and X^+ coincide, and 2) the *Schur property* or *X respects sequentiality* if X and X^+ have the same set of convergent sequences. Clearly, if X respects compactness, then it has the Schur property.

Let $X \in \mathbf{LCA}$ and let \mathcal{P} be a topological property. The question of whether X respects \mathcal{P} has been intensively studied for many natural properties \mathcal{P} . The famous Glicksberg theorem [40] states that every LCA group X respects compactness, and hence X has the Schur property. So the Glicksberg and the Schur properties are respected properties in **LCA**. Trigos-Arrieta [66, 67] proved that also pseudocompactness and functional boundedness are respected properties in **LCA**. Banaszczyk and Martín-Peinador [9] generalized these results to the class **Nuc** of all nuclear groups. Hernández, Galindo and Macario [46] characterized Banach spaces which have the Schur property. Recently Außenhofer [2] proved that every locally quasi-convex Schwartz group respects compactness. These results lead us to the next question which is also considered in the article:

Problem 1.5. *Let F be a functor in **MAPA** and let \mathcal{P} be a respected property in a subcategory \mathbf{A} of **MAPA**. Characterize $X \in \mathbf{A}$ such that $F(X)$ respects \mathcal{P} .*

(D) *The functor \mathfrak{F}_0 in **TAG**.* Now we define the functor \mathfrak{F}_0 in **TAG** introduced by Dikranjan, Martín-Peinador and Tarieladze in [23]. Let X be an Abelian topological group and let $\mathcal{N}(X)$ be the filter of all open neighborhoods at zero in X . Denote by $X^{\mathbb{N}}$ the group of all sequences $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$. The subgroup of $X^{\mathbb{N}}$ of all sequences eventually equal to zero we denote by $X^{(\mathbb{N})}$. It is easy to check that the collection $\{V^{\mathbb{N}} : V \in \mathcal{N}(X)\}$ forms a base at 0 for a group topology in $X^{\mathbb{N}}$. This topology is called *the uniform topology* and is denoted by \mathbf{u} [23]. Following [23], denote by $c_0(X)$ the following subgroup of $X^{\mathbb{N}}$

$$c_0(X) := \left\{ (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} : \lim_n x_n = 0 \right\}.$$

The uniform group topology on $c_0(X)$ induced from $(X^{\mathbb{N}}, \mathbf{u})$ we denote by \mathbf{u}_0 . Following [23], define the functor \mathfrak{F}_0 on the category **TAG** by the assignment

$$X \rightarrow \mathfrak{F}_0(X) := (c_0(X), \mathbf{u}_0).$$

If $X = \mathbb{R}$, then $\mathfrak{F}_0(\mathbb{R})$ coincides with the classical Banach space c_0 . The groups of the form $\mathfrak{F}_0(X)$ were thoroughly studied in [23, 36, 37]. In particular, \mathfrak{F}_0 is finitely multiplicative [36].

Clearly, for every natural number n , the group X^n is both a direct summand and a quotient group of $\mathfrak{F}_0(X)$. So, if $\mathfrak{F}_0(X)$ has a topological property of one of the types \mathcal{P}_s or \mathcal{P}_q , then X^n has the same property for every $n \in \mathbb{N}$. Thus, if \mathfrak{F}_0 preserves a property $\mathcal{P} \in \mathcal{P}_s \cup \mathcal{P}_q$, then \mathcal{P} must be preserved by any finite cartesian power of X .

The functor \mathfrak{F}_0 preserves many important topological properties.

Fact 1.6. *Let $X \in \mathbf{TAG}$. Then*

- (i) ([23, 3.4]) *Let \mathcal{P} denote one of the following topological properties from $\mathcal{P}_s \cup \mathcal{P}_q$: completeness, sequential completeness, metrizability, separability, maximal almost periodicity, local quasi-convexity and connectedness. Then $\mathfrak{F}_0(X)$ has \mathcal{P} if and only if X has \mathcal{P} .*
- (ii) ([23, 3.1]) *$\mathfrak{F}_0(X)$ is precompact if and only if $X = \{0\}$.*
- (iii) ([37]) *$\mathfrak{F}_0(X)$ is locally precompact if and only if X is discrete.*

Let $\mathbf{u} = \{u_n\}_{n \in \omega}$ be a sequence in the dual group \widehat{X} of an Abelian topological group X and let H be a subgroup of X . Following [25], we set

$$s_{\mathbf{u}}(X) := \{x \in X : (u_n, x) \rightarrow 1\},$$

and say that a subgroup H is \mathfrak{g} -closed in X if

$$H = \bigcap_{\mathbf{u} \in \widehat{X}^{\mathbb{N}}} \{s_{\mathbf{u}}(X) : H \leq s_{\mathbf{u}}(X)\}.$$

Some properties of the closure operator \mathfrak{g} are given in [21].

The question of whether $c_0(X)$ is a \mathfrak{g} -closed subgroup of $X^{\mathbb{N}}$ was considered in [36]. For the groups with the Schur property we have:

Fact 1.7. ([36]) *Let X be a MAP Abelian group. If X has the Schur property, then $c_0(X)$ is a \mathfrak{g} -closed subgroup of $X^{\mathbb{N}}$.*

Below (see Theorem 1.11) we show that in general the \mathfrak{g} -closeness of $c_0(X)$ in $X^{\mathbb{N}}$ does not imply that X has the Schur property.

Our interest in the study of the functor \mathfrak{F}_0 is motivated also by the following arguments. Firstly, if X is metrizable and connected compact Abelian group, then the Bohr modification of $\mathfrak{F}_0(X)$ is a connected precompact metrizable non-Mackey group [23] (the definition of Mackey groups and their basic properties see [13]). Secondly, Rolewicz [58] observed that the complete metrizable group $\mathfrak{F}_0(\mathbb{T})$ is monothetic (see a proof of this fact in [27, pp. 20-21]). So monothetic Polish groups need not be compact nor discrete. In particular, these two arguments show that the groups of the form $\mathfrak{F}_0(X)$ represent a nice source of (counter)examples in different areas of topological algebra. Further, the group $\mathfrak{F}_0(\mathbb{T})$ plays a significant role in the theory of characterized subgroups of compact Abelian groups (see [32, 35]). The next result especially emphasizes the importance of the study of the functor \mathfrak{F}_0 :

Fact 1.8. ([37]) *For every LCA group X , the group $\mathfrak{F}_0(X)$ is reflexive.*

So the functor \mathfrak{F}_0 preserves reflexivity on the class \mathbf{LCA} . On the other hand, excepting the case of discrete X , $\mathfrak{F}_0(X)$ is never locally precompact by Fact 1.6(iii). Hence we obtain a big class of reflexive complete Abelian groups which is beyond \mathbf{LCA} .

The main results. We emphasize that the study of the functor \mathfrak{F}_0 , as well as other functors defined in a reasonable way in \mathbf{TAG} , in the light of Problems 1.1-1.5 is important not only by the natural categorical reasons, but also because we obtain in this way a *class* of groups with or without given topological properties.

The main goal of the article is the study of both important topological properties preserved under \mathfrak{F}_0 (Theorems 1.9 and 1.11), and the preservation under \mathfrak{F}_0 of respected properties in the class \mathbf{LCA} (Theorem 1.10).

In the first main result we complete the list of general topological properties which are preserved under the functor \mathfrak{F}_0 stated in Fact 1.6, see Problems 1.1 and 1.2 (all relevant definitions are given in Section 3).

Theorem 1.9. *Let X be an Abelian topological group. Then:*

- (i) *X is an (E) -space if and only if $\mathfrak{F}_0(X)$ is an (E) -space.*
- (ii) *X is a strictly angelic space if and only if $\mathfrak{F}_0(X)$ is strictly angelic.*
- (iii) *X is a \check{S} -space if and only if $\mathfrak{F}_0(X)$ is a \check{S} -space.*

On the other hand, under additional hypothesis and taking into account that \mathfrak{F}_0 contains $X \times X$ as a closed subgroup, the functor \mathfrak{F}_0 does not preserve countable compactness, sequentiality and Fréchet-Urysohnness (see Remark 3.5 below). Further, since every pseudocompact group is precompact [1], $\mathfrak{F}_0(X)$ is pseudocompact if and only if $X = \{0\}$ by Fact 1.6(ii).

In the next theorem we give a complete answer to Problem 1.5 for $F = \mathfrak{F}_0$ and various properties in $\mathbf{A} = \mathbf{LCA}$.

Theorem 1.10. *Let X be a LCA group. Then the following are equivalent:*

- (i) X is totally disconnected.
- (ii) $\mathfrak{F}_0(X)$ embeds into the product of a family of LCA groups.
- (iii) $\mathfrak{F}_0(X)$ is a nuclear group.
- (iv) $\mathfrak{F}_0(X)$ is a Schwartz groups.
- (v) $\mathfrak{F}_0(X)$ respects compactness.
- (vi) $\mathfrak{F}_0(X)$ has the Schur property.
- (vii) $\mathfrak{F}_0(X)$ respects countable compactness.
- (viii) $\mathfrak{F}_0(X)$ respects sequential compactness.
- (ix) $\mathfrak{F}_0(X)$ respects pseudocompactness.
- (x) $\mathfrak{F}_0(X)$ respects functional boundedness.

Moreover, if E is a functionally bounded subset in $\mathfrak{F}_0(X)^+$, then its closure $\text{cl}_{\mathfrak{F}_0(X)}(E)$ is compact in $\mathfrak{F}_0(X)$.

Concerning the equivalence of items (ii) and (v) we note that, Remus and Trigos-Arrieta [56] found Abelian topological groups which cannot be embedded into products of LCA groups though they are reflexive and respect compactness.

Now we formulate the third main theorem of the article:

Theorem 1.11. *Let X be a compact connected metrizable Abelian group. Then:*

- (i) $\mathfrak{F}_0(X)$ is a monothetic connected Polish non-Schwartz Abelian group.
- (ii) $\mathfrak{F}_0(X)$ has no the Schur property, and hence it does not respect compactness.
- (iii) $c_0(\mathfrak{F}_0(X))$ is \mathfrak{g} -closed in $\mathfrak{F}_0(X)^{\mathbb{N}}$.

In the next remark we comment Theorem 1.11:

Remark 1.12. (1) In item (i) we generalize the above-mentioned result of Rolewicz [58]. Note that our proof differs from Rolewicz's one even for $X = \mathbb{T}$.

(2) Remus and Trigos-Arrieta in [55] provided examples of reflexive Abelian groups which do not respect compactness. They gave counterexamples to a result published by Venkataraman in [69]. However a stronger result was known. Namely, the space c_0 is reflexive by [62] and does not have even the Schur property (see [20, Exercise 3 on p. 212]). Items (i) and (ii) and Fact 1.8 show that there exists reflexive monothetic Polish groups which do not respect sequentiality.

(3) In connection with Fact 1.7 we ask in Problem 49 of [36]: does \mathfrak{g} -closeness of $c_0(X)$ in $X^{\mathbb{N}}$ imply that X has the Schur property? The items (ii) and (iii) and Fact 1.8 show that in general the answer to this problem is negative even for reflexive Polish groups.

The article is organized as follows. In Section 2 we formulate and prove some important auxiliaries results which are used essentially in the article. We prove Theorem 1.9 in Section 3. In Section 4 we discuss and prove some general results concerning respected properties in topological groups. In particular, we introduce a new property (the SB -property) which gives a sufficient condition for topological groups to have the Glicksberg property etc. (see Theorem 4.6). Theorem 1.10 is proved in Section 5, and Theorem 1.11 we prove in Section 6. In Section 7 we show that the Glicksberg property and the Schur property can be naturally defined by some functors in \mathbf{TAG} . In the last section we define another functors naturally coming from Functional Analysis and pose a dozen open problems.

2. AUXILIARY RESULTS

2.1. Uniformly discrete and bounded subsets in topological groups. The closure of a subset E of a topological space Y we denote by \overline{E} or $\text{cl}_Y(E)$.

Let X be a topological group and $U \in \mathcal{N}(X)$. Recall that a subset A of X is called *left (right) U -separated* if $aU \cap bU = \emptyset$ (respectively, $Ua \cap Ub = \emptyset$) for every distinct elements $a, b \in A$. A subset E of X is called *left (right) uniformly discrete* if there is $U \in \mathcal{N}(X)$ such that E is left (right) U -separated.

We omit proofs of the next two simple lemmas.

Lemma 2.1. *Every left (right) uniformly discrete subset A of a topological group X is closed.*

Lemma 2.2. *Let E be a subset of a topological group X . Then, for every $U \in \mathcal{N}(X)$, there exists a maximal (under inclusion) left (right) U -separated subset of E .*

Recall that, a subset E of a topological group X is called *left-precompact* (respectively, *right-precompact*, *precompact*) if, for every $U \in \mathcal{N}(X)$, there exists a finite subset F of X such that $E \subseteq F \cdot U$ (respectively, $E \subseteq U \cdot F$, $E \subseteq F \cdot U$ and $E \subseteq U \cdot F$). If E is symmetric the three different definitions coincide.

In the sequel we essentially use the next useful proposition which holds true for every uniform space (see [5]). For the sake of completeness we prove it.

Proposition 2.3. *Let E be an infinite subset of a topological group X . Then E is either left (right) precompact or it has a countably infinite left (right) uniformly discrete subset.*

Proof. Assume that E contains a countably infinite left uniformly discrete subset A . Let us show that E is not left precompact. Suppose for a contradiction that A is left precompact. Take a symmetric $U \in \mathcal{N}(X)$ such that A is left $(U \cdot U)$ -separated. As E is left precompact, there is a finite subset F of X such that $E \subseteq F \cdot U$. In particular, for every $a \in A$, there are $f_a \in F$ and $u_a \in U$ such that $a = f_a u_a$.

We claim that $f_a \neq f_b$ for every distinct $a, b \in A$. Indeed, otherwise, $a^{-1}b = (f_a u_a)^{-1}(f_a u_b) \in U \cdot U$. Hence A is not left $(U \cdot U)$ -separated, a contradiction. This means that the map $a \mapsto f_a$ is injective. As F is finite, we obtain that also A is finite. This contradiction shows that E is not left precompact.

Let now E do not have countably infinite left uniformly discrete subsets. We claim that E is left precompact. Indeed, let $U \in \mathcal{N}(X)$. Take a symmetric $V \in \mathcal{N}(X)$ such that $V \cdot V \subseteq U$. By Lemma 2.2, choose a maximal left V -separated subset F in E . By assumption F is finite. Let us show that $E \subseteq F \cdot U$. Suppose for a contradiction that there is $g \in E \setminus (F \cdot U)$. Then, for every $f \in F$, we have $g \cdot V \cap f \cdot V = \emptyset$ since otherwise, $g \in f(V \cdot V) \subseteq F \cdot U$. This means that the set $F' := \{g\} \cup F$ is left V -separated that contradicts the maximality of F to be a left V -separated subset of E . So $E \subseteq F \cdot U$. Thus E is left precompact. \square

Recall (see [63]) that a subset E of a topological space Y is called *functionally bounded in Y* if every continuous real-valued function on Y is bounded on E . The family of all functionally bounded subsets of Y we denote by $\mathcal{FB}(Y)$. We will use the following trivial fact:

Lemma 2.4. *Let Y and Z be topological spaces and let $f : Y \rightarrow Z$ be a continuous mapping. Then:*

- (1) *The family $\mathcal{FB}(Y)$ contains all compact subsets of Y and it is closed under taking closure, finite unions and arbitrary subsets.*
- (2) *$f(\mathcal{FB}(Y)) \subseteq \mathcal{FB}(Z)$.*

We prove the next lemma for the sake of completeness.

Lemma 2.5. *Let A be a left (right) uniformly discrete subset of a topological group X . Then A is functionally bounded if and only if it is finite.*

Proof. If A is finite it is trivially functionally bounded.

Assume now that A is functionally bounded in X . Let A be left $(U \cdot U)$ -separated for a symmetric $U \in \mathcal{N}(X)$. Take a continuous function $f : X \rightarrow [0, 1]$ such that $f(e) = 1$ and $f|_{X \setminus U} = 0$ [48, 8.4]. Suppose

for a contradiction that A is infinite. Choose arbitrarily a one-to-one sequence $\{a_n\}_{n \in \mathbb{N}}$ in A and set

$$g(x) := \sum_{n \in \mathbb{N}} n \cdot f(a_n^{-1} \cdot x), \quad \forall x \in X.$$

Note that $g(x)$ is well-defined since $f(a_n^{-1} \cdot x) \neq 0$ only if $x \in a_n U$. Further, for every $x \in X$, the set xU intersects with $a_n U$ for at most one $n \in \mathbb{N}$. So $g(x)$ is continuous on X and unbounded on A . Hence A is not functionally bounded in X , a contradiction. Thus A is finite. \square

As a corollary of Proposition 2.3 and Lemma 2.5 we obtain:

Proposition 2.6. ([61, Statement A]) *Every functionally bounded subset E of a topological group X is precompact.*

Proof. By Lemmas 2.4(1) and 2.5, the set E does not have infinite left (right) uniformly discrete subsets. Now Proposition 2.3 implies that E is left and right precompact. Thus E is precompact. \square

In what follows we use the next fact:

Fact 2.7. ([1, 3.7.10]) *Let X be a (Raïkov) complete topological group. Then a subset A of X is precompact if and only if its closure \overline{A} is compact.*

Proposition 2.6 and Fact 2.7 immediately implies:

Corollary 2.8. *Let E be a subset of a complete topological group X . Then E is functionally bounded iff E is precompact iff the closure \overline{E} of E is compact.*

2.2. The Bohr topology on MAP groups. For a subset A of an Abelian topological group X , the annihilator of A in \widehat{X} is $A^\perp := \{u \in \widehat{X} : (u, x) = 1, \forall x \in A\}$. The group \widehat{X} endowed with the compact-open topology we denote by X^\wedge .

Recall that a subgroup H of an Abelian topological group X is called *dually closed* in X if for every $x \in X \setminus H$ there exists a character $\chi \in H^\perp$ such that $(\chi, x) \neq 1$, i.e., $H = \bigcap_{\chi \in H^\perp} \ker(\chi)$. Clearly, every dually closed subgroup is closed. A subgroup H is called *dually embedded* in X if every continuous character of H can be extended to a continuous character of X .

Fact 2.9. ([54, Lemma 3.3]) *Every open subgroup of an Abelian topological group X is dually closed and dually embedded in X .*

Note that reflexive groups may contain closed non-dually closed subgroups, see [54, Example (ii)].

An Abelian group X endowed with the discrete topology we denote by X_d . If H is a dense subgroup of $(X_d)^\wedge$, we denote by $\sigma(X, H)$ or T_H the weakest group topology on X for which the elements of H are continuous. In particular, for each $X \in \mathbf{MAPA}$ the sets of the form

$$\{x \in X : (\chi, x) \in \mathbb{T}_+\},$$

where $\chi \in \widehat{X}$ and $\mathbb{T}_+ := \{z \in \mathbb{T} : \operatorname{Re}(z) \geq 0\}$, form a subbase of the weak topology $\sigma(X, \widehat{X})$ on X .

Fact 2.10. ([16]) *Let X be an Abelian group and let H_1 and H_2 be dense subgroups of $(X_d)^\wedge$. Then $T_{H_1} \subseteq T_{H_2}$ if and only if $H_1 \subseteq H_2$. Hence $T_{H_1} = T_{H_2}$ iff $H_1 = H_2$.*

The next lemma generalizes some well-known results for LCA groups, see for example [66, 67]. Item (1) and the necessity in item (4) see in [21] and [2, 2.1]. Below we generalize this lemma to non-Abelian MAP groups. For the convenience of the reader we give its complete proof.

Lemma 2.11. *Let H be a subgroup of a MAP Abelian group X . Then*

- (1) ([21]) *H is dually closed in X if and only if H is closed in X^+ .*
- (2) *If H is a dually closed subgroup of X , then $(X/H)^+ = X^+/H$.*
- (3) $\sigma(X, \widehat{X})|_H \leq \sigma(H, \widehat{H})$.

(4) ([45]) H is dually embedded in X if and only if $H^+ = \left(H, \sigma(X, \widehat{X})|_H \right)$.

(5) H is dually closed and dually embedded in X if and only if H^+ is a closed subgroup of X^+ .

Proof. (1) Let H be dually closed in X . Then $H = \bigcap_{\chi \in H^\perp} \ker(\chi)$. So H is closed in $\sigma(X, \widehat{X})$.

Conversely, let H be closed in $\sigma(X, \widehat{X})$ and $x \in X \setminus H$. Denote by $q^+ : X^+ \rightarrow X^+/H$ the quotient map. Then X^+/H is a precompact group and $q^+(x) \neq 0$. It is well-known that there exists $\eta \in (X^+/H)^\wedge$ such that $(\eta, q^+(x)) \neq 1$. So $\eta \circ q^+ \in \widehat{X^+} = \widehat{X}$ and $(\eta \circ q^+, x) = (\eta, q^+(x)) \neq 1$. Thus H is dually closed in X .

(2) By item (1), H is a closed subgroup of X^+ . Hence X/H and X^+/H are well-defined. Let $i : X/H \rightarrow X^+/H$ be the identity continuous isomorphism and let $q : X \rightarrow X/H$ be the quotient map. By Fact 2.10, we have to show that $\widehat{X/H} = \widehat{X^+/H}$. Since i is continuous we have $\widehat{X^+/H} \subseteq \widehat{X/H}$.

Let us prove the converse inclusion. Fix $\eta \in \widehat{X/H}$. Then $\eta \circ q \in \widehat{X}$. So $\eta \circ q \in \widehat{X^+}$. Since $\eta \circ q|_H = 0$ and H is closed in X^+ , we obtain $\eta \in \widehat{X^+/H}$. Thus $\widehat{X/H} \subseteq \widehat{X^+/H}$. Therefore, $\widehat{X/H} = \widehat{X^+/H}$ and $(X/H)^+ = X^+/H$.

(3) Since $\chi|_H \in \widehat{H}$ for every $\chi \in \widehat{X}$, the assertion follows from Fact 2.10.

(4) Assume that H is dually embedded in X and $\eta \in \widehat{H}$. Denote by χ an arbitrary extension of η to a continuous character of X . Then

$$\{h \in H : (\eta, h) \in \mathbb{T}_+\} = H \cap \{x \in X : (\chi, x) \in \mathbb{T}_+\}.$$

So $\sigma(H, \widehat{H}) \subseteq \sigma(X, \widehat{X})|_H$. The converse inclusion follows from item (3). Thus $\sigma(H, \widehat{H}) = \sigma(X, \widehat{X})|_H$. Therefore, $H^+ = \left(H, \sigma(X, \widehat{X})|_H \right)$.

Let now $H^+ = \left(H, \sigma(X, \widehat{X})|_H \right)$. So $\sigma(H, \widehat{H}) = \sigma(X, \widehat{X})|_H = \sigma(H, \widehat{X}|_H)$. It follows from Fact 2.10 that $\widehat{H} = \widehat{X}|_H$. This explicitly means that every $\eta \in \widehat{H}$ can be extended to $\chi \in \widehat{X}$. Thus H is dually embedded in X .

(5) follows from items (1) and (4). □

The next lemma extends Lemma 4.4 in [67].

Lemma 2.12. *Let H be an open subgroup of a MAP Abelian group X . If E is a functionally bounded subset in X^+ , then E is contained in a finite union of cosets of H .*

Proof. By Fact 2.9, H is dually closed in X . Hence H is closed in X^+ and $(X/H)^+ = X^+/H$ by Lemma 2.11. Let $\pi : X^+ \rightarrow X^+/H$ be the quotient map. Then $\pi(E)$ is functionally bounded subset in X^+/H (see Lemma 2.4). We have to show that $\pi(E)$ is finite. As X/H is discrete and $\pi(E)$ is functionally bounded in $(X/H)^+$, $\pi(E)$ is finite by [67, Lemma 4.4]. □

Now we consider the Bohr compactification of non-Abelian groups. Let X be a Hausdorff topological group. Recall that a compact group bX is called the *Bohr compactification* of X if there exists a continuous homomorphism i from X onto a dense subgroup of bX such that the pair (bX, i) satisfies the following *universal property*: If $p : X \rightarrow C$ is a continuous homomorphism into a compact group C , then there exists a continuous homomorphism $j^p : bX \rightarrow C$ such that $p = j^p \circ i$. Following von Neumann [52], the group X is called *maximally almost periodic* (MAP) if the group X^+ is Hausdorff, where X^+ is the group X endowed with the topology induced from bX . The family **MAP** of all MAP topological groups is a subcategory of the category **TG** of all Hausdorff topological groups and continuous homomorphisms.

Note that every irreducible representation of a compact group is finite-dimensional (see [48, 22.13]). It is well-known also that we can identify the set of all finite-dimensional irreducible representations of a MAP Abelian group X with the usual set of its continuous characters.

Let X be a MAP group. Denote by \widehat{X} the set of all finite-dimensional irreducible representations of X . For a finite-dimensional irreducible representation $u \in \widehat{X}$ of X by unitary operators on a Hilbert space \mathcal{H} and an arbitrary $x \in X$ we denote by (u, x) the operator $u(x)$ on the representation space \mathcal{H} of u . The

identity operator on \mathcal{H} is denoted by $I_{\mathcal{H}}$ (or just by I). The next folklore lemma easily follows from the definition of Bohr compactification and irreducible representation and Corollary 3.6.17 of [1] (we denote by \overline{X} the completion of X):

Lemma 2.13. *If X is a MAP group, then $b\overline{X} = bX$ and $\widehat{X} = \widehat{\overline{X}} = \widehat{X^+} = \widehat{bX}$.*

Next fact immediately follows from the definition of the Bohr compactification (cf. [9]):

Fact 2.14. *Let $\phi : X \rightarrow Y$ be a continuous homomorphism between MAP groups. Then $\phi : X^+ \rightarrow Y^+$ is also continuous.*

Note that the definitions of the dual closure and the dual embedding are also transferred to non-Abelian case without modifications. A subgroup H of a MAP group X is called *dually closed* in X if for every $g \in X \setminus H$ there exists an irreducible (finite-dimensional) representation $\chi \in H^\perp := \{\eta \in \widehat{X} : \eta|_H = I\}$ such that $(\chi, g) \neq I$ ([33]). And H is named *dually embedded* in X if every $\chi \in \widehat{H}$ can be extended to an irreducible (finite-dimensional) representation of X ([45]). The next lemma generalizes items (1), (4) and (5) of Lemma 2.11:

Lemma 2.15. *Let H be a subgroup of a MAP group X and $p : H \rightarrow X$ be the natural embedding. Then*

- (1) *H is dually closed in X if and only if H is a normal subgroup of X and H is closed in X^+ .*
- (2) *([45]) H is dually embedded in X if and only if $j^p|_H : H^+ \rightarrow X^+$ is an embedding.*
- (3) *H is dually closed and dually embedded in X if and only if H^+ is a closed subgroup of X^+ .*

Proof. (1) Let H be dually closed in X . Then $H = \bigcap_{\chi \in H^\perp} \ker(\chi)$. So H is a normal subgroup of X and it is closed in X^+ .

Conversely, let H be closed and normal in X^+ and let $x \in X \setminus H$. Denote by $q^+ : X^+ \rightarrow X^+/H$ the quotient map. Then X^+/H is a precompact group and $q^+(x) \neq e$. Since X^+/H has a compact completion [1, 3.7.16], Peter-Weyl's theorem implies that there exists $\eta \in \widehat{X^+/H}$ such that $(\eta, q^+(x)) \neq I$. So, by Lemma 2.13, $\eta \circ q^+ \in \widehat{X^+} = \widehat{X}$ and $(\eta \circ q^+, x) = (\eta, q^+(x)) \neq I$. Thus H is dually closed in X .

(3) follows from items (1) and (2). □

In what follows we use the next property:

Definition 2.16. *A topological group X is said to have an **sp**-property if every separable precompact subset of X has compact closure.*

If a topological group X is complete, the closure \overline{E} of each precompact subset E is compact by Fact 2.7. So we obtain:

Proposition 2.17. *Every complete topological group has the **sp**-property.*

Recall that a (Hausdorff) topological space Y is called *sequentially compact* if every sequence in Y contains a convergent subsequence.

Example 2.18. There is a sequentially compact non-compact Abelian group H which has the **sp**-property. We use the notion of Σ -product introduced by Pontryagin. Let $G := X^\kappa$, where X is a metrizable compact Abelian group and the cardinal κ is uncountable. For $g = (x_i)_{i \in \kappa} \in G$, denote $\text{supp}(g) := \{i \in \kappa : x_i \neq 0\}$ and set

$$H := \{g \in G : |\text{supp}(g)| \leq \aleph_0\}.$$

Then H with the induced topology is a dense subgroup of G . We claim that H is a sequentially compact group with the **sp**-property. Indeed, any countable subset of H is contained in a countable product Y of copies of X . Since Y is a compact and metrizable subgroup of X , we obtain that the group H is sequentially compact with the **sp**-property. Note also that H is Fréchet-Urysohn by [53].

2.3. Some results on groups of the form $\mathfrak{F}_0(X)$. Let X be an Abelian topological group. For every $n \in \mathbb{N}$, define an injective homomorphism $\nu_n : X \rightarrow X^{(\mathbb{N})}$ by

$$\nu_n(x) = (0, \dots, 0, x, 0 \dots),$$

where $x \in X$ is placed in position n . Define the projections $p_n : X^{\mathbb{N}} \rightarrow X^n$ and $\pi_n : X^{\mathbb{N}} \rightarrow X$ by

$$p_n(\mathbf{x}) := (x_1, \dots, x_n), \quad \text{and} \quad \pi_n(\mathbf{x}) := x_n, \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}.$$

Clearly, p_n and π_n are continuous in the uniform topology \mathbf{u} . We use the same notations for the restrictions of p_n and π_n onto $c_0(X)$.

Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of an Abelian topological group X . Following [36], we say that $\{E_n\}$ is a *null-sequence* in X if, for every $U \in \mathcal{N}(X)$, $E_n \subseteq U$ for all sufficiently large $n \in \mathbb{N}$.

Lemma 2.19. *If $\{E_n\}_{n \in \mathbb{N}}$ is a null-sequence of subsets in an Abelian topological group X , then $\{\overline{E_n}\}_{n \in \mathbb{N}}$ is a null-sequence in X as well.*

Proof. Let $U \in \mathcal{N}(X)$. Take $V \in \mathcal{N}(X)$ such that $\overline{V} \subseteq U$ [48, 4.7]. Choose $m \in \mathbb{N}$ such that $E_n \subseteq V$ for every $n \geq m$. Then $\overline{E_n} \subseteq \overline{V} \subseteq U$ for every $n \geq m$. Thus $\{\overline{E_n}\}_{n \in \mathbb{N}}$ is a null-sequence in X . \square

Lemma 2.20. *Let E be a subset of $\mathfrak{F}_0(X)$ for an Abelian topological group X . If $\{\pi_n(E)\}_{n \in \mathbb{N}}$ is not a null-sequence in X , then E has an infinite uniformly discrete subset.*

Proof. By assumption, there exists $U \in \mathcal{N}(X)$ such that $\pi_n(E) \not\subseteq U$ for an infinite set of indices. Choose a symmetric $V \in \mathcal{N}(X)$ such that $V + V \subseteq U$. We shall build a uniformly discrete sequence in E by induction.

Take $n_1 \in \mathbb{N}$ such that $\pi_{n_1}(E) \not\subseteq U$, and choose $\mathbf{b}_1 = (b_n^1)_{n \in \mathbb{N}} \in E$ such that $\pi_{n_1}(\mathbf{b}_1) = b_{n_1}^1 \notin U$. Since $\mathbf{b}_1 \in c_0(X)$, there is an index $j_1, j_1 > n_1$, such that $b_i^1 \in V$ for every $i \geq j_1$.

Take $n_2 \in \mathbb{N}, n_2 > j_1$, such that $\pi_{n_2}(E) \not\subseteq U$, and choose $\mathbf{b}_2 = (b_n^2)_{n \in \mathbb{N}} \in E$ such that $\pi_{n_2}(\mathbf{b}_2) = b_{n_2}^2 \notin U$. Since $\mathbf{b}_2 \in c_0(X)$, there is an index $j_2, j_2 > n_2$, such that $b_i^2 \in V$ for every $i \geq j_2$.

Continuing this process we shall build a sequence $\{\mathbf{b}_n\}_{n \in \mathbb{N}}$ in E . If $k < m$, then $\pi_{n_m}(\mathbf{b}_m - \mathbf{b}_k) = b_{n_m}^m - b_{n_m}^k \notin V$ since, otherwise, $b_{n_m}^m \in b_{n_m}^k + V \subset V + V \subseteq U$ that contradicts the choice of \mathbf{b}_m . Thus the sequence $\{\mathbf{b}_n\}$ is $V^{\mathbb{N}} \cap c_0(X)$ -separated. \square

The next two results play an essential role in the sequel:

Fact 2.21. ([36]) *Let X and Y be Abelian topological groups.*

- (1) *The groups $\mathfrak{F}_0(X) \times \mathfrak{F}_0(Y)$ and $\mathfrak{F}_0(X \times Y)$ are topologically isomorphic.*
- (2) *If H is a (respectively, closed or open) subgroup of X , then $\mathfrak{F}_0(H)$ is a (respectively, closed or open) subgroup of $\mathfrak{F}_0(X)$.*
- (3) *A closed subset K of $\mathfrak{F}_0(X)$ is compact if and only if the sequence $\{\pi_n(K)\}_{n \in \mathbb{N}}$ is a null-sequence of compact subsets of X . Moreover, if K is compact, then the product $\prod_{n \in \mathbb{N}} \pi_n(K)$ is a compact subset of $\mathfrak{F}_0(X)$.*

Fact 2.22. ([37]) *Let X be a LCA group and $X \cong \mathbb{R}^n \times X_0$, where $n \in \omega$ and X_0 has an open compact subgroup. Then*

- (1) $\widehat{\mathfrak{F}_0(X)} \cong c_0^n \times \widehat{\mathfrak{F}_0(X_0)}$ and $\widehat{\mathfrak{F}_0(X)}^\wedge \cong \ell_1^n \times \widehat{\mathfrak{F}_0(X_0)}^\wedge$.
- (2) $\widehat{\mathfrak{F}_0(X_0)} \subseteq \widehat{X_0}^{\mathbb{N}}$. Further, $\mathbf{g} = (g_n)_{n \in \mathbb{N}} \in \widehat{\mathfrak{F}_0(X_0)}^\wedge$ if and only if there exists an open subgroup H of X_0 and a natural number m such that $g_n \in H^\perp$ for every $n \geq m$. In this case we have

$$(\mathbf{g}, \mathbf{x}) = \lim_{n \rightarrow \infty} \prod_{i=1}^n (g_i, x_i), \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \widehat{\mathfrak{F}_0(X)}.$$

3. TOPOLOGICAL PROPERTIES WHICH ARE PRESERVED UNDER THE FUNCTOR \mathfrak{F}_0

Let X be a topological space. Recall that a subset A of X is called

- *relatively compact* if its closure \bar{A} is compact;
- *relatively countably compact* if each countably infinite subset in A has a cluster point in X ;
- *relatively sequentially compact* if each sequence in A has a subsequence converging to a point of X ;
- *countably compact* or, respectively, *sequentially compact* if in the above two definitions the cluster point or, respectively, the limit point is required to be in A .

Clearly, compact and sequentially compact subsets are countably compact. On the other hand, there are sequentially compact subsets of a completely regular Hausdorff space which are nonclosed, and their closure are not countably compact (see Examples 28 and 29 in [11]). So topological spaces, in which any two of these in general different topological properties coincide, are of independent interest. Let us recall some of them.

A Hausdorff topological space X is called

- (1) an *(E)-space* if its relatively countably compact subsets are relatively compact (see [42, Exercise 1, p. 209]);
- (2) a *\check{S} -space* if its compact subsets are sequentially compact;
- (3) an *angelic space* if for every relatively countably compact subset A of X the following two claims hold:
 - (i) A is relatively compact, and
 - (ii) if $x \in \bar{A}$, then there is a sequence in A which converges to x ;
- (4) a *strictly angelic space* if it is angelic and each its separable compact subspace is first countable ([41]).

It is easy to see that all classes of spaces (1)-(4) are closed under taking closed subspaces, i.e., they are of \mathcal{P}_s -type. Note that the product of two countably compact spaces may not be countably compact [31, 3.10.19]. On the other hand, the countable product of (sequentially) compact spaces is (sequentially) compact (see [31, 3.10.35]). Also the class of strictly angelic spaces is countably productive [41]. This explains why Theorem 1.9 is valid in the case when countable compactness coincides with (sequential) compactness. Clearly, every (strictly) angelic space is an *(E)-space*.

Now we are in position to prove Theorem 1.9 (we use notations from Section 2.3).

Proof of Theorem 1.9. Since all the classes of groups (i)-(iii) in the theorem are closed under taking closed subgroups, we need to prove only the necessity in (i)-(iii).

(i) Assume that X is an *(E)-space*. We have to show that $\mathfrak{F}_0(X)$ is also an *(E)-space*. To end this we have to prove that the closure \bar{A} of every relatively countably compact subset A of $\mathfrak{F}_0(X)$ is compact.

For every $n \in \mathbb{N}$, set $A_n := \pi_n(A)$. We claim that A_n is relatively countably compact in X . Indeed, for every countably infinite subset $\{a_k^n\}_{k \in \omega}$ in A_n take arbitrarily its preimage $\{\mathbf{a}_k\}_{k \in \omega}$ in A , i.e., $\pi_n(\mathbf{a}_k) = a_k^n$ for every $k \in \omega$. By definition, $\{\mathbf{a}_k\}_{k \in \omega}$ has a cluster point \mathbf{a} in $\mathfrak{F}_0(X)$. Clearly, $\pi_n(\mathbf{a})$ is a cluster point of A_n .

Since X is an *(E)-space*, we obtain that the closure $\overline{A_n}$ of A_n is compact in X for every $n \in \mathbb{N}$.

Let us show that the sequence $\{A_n\}_{n \in \omega}$ is a null sequence in X . Indeed, otherwise A has a uniformly discrete sequence $\{\mathbf{b}_k\}_{k \in \mathbb{N}}$ by Lemma 2.20. Since the set $\{\mathbf{b}_k\}$ is closed in $\mathfrak{F}_0(X)$ by Lemma 2.1, it does not have cluster points. Thus A is not relatively countably compact, a contradiction.

Lemma 2.19 implies that $\{\overline{A_n}\}_{n \in \mathbb{N}}$ is a null sequence of compact subsets of X . Set $K := \prod_{n \in \mathbb{N}} \overline{A_n}$. Fact 2.21(3) yields that K is a compact subset of $\mathfrak{F}_0(X)$. Since $A \subseteq K$ we obtain that \bar{A} is compact in $\mathfrak{F}_0(X)$. Thus $\mathfrak{F}_0(X)$ is an *(E)-space*.

(ii) Assume that X is a strictly angelic space. We have to show that $\mathfrak{F}_0(X)$ is strictly angelic as well.

Let us show first that every separable compact subset K of $\mathfrak{F}_0(X)$ is first countable. For every $n \in \mathbb{N}$, set $K_n := \pi_n(K)$ and put $K' := \prod_{n \in \mathbb{N}} K_n$. By Fact 2.21(3), K' is a compact subset of $\mathfrak{F}_0(X)$. Since X is

strictly angelic, K_n is first countable for every $n \in \mathbb{N}$. Thus K' and its compact subset K are first countable (see [31, 2.3.14]).

Let B be a relatively countably compact subset of $\mathfrak{F}_0(X)$. By item (i), B is relatively compact in $\mathfrak{F}_0(X)$. So, to prove that $\mathfrak{F}_0(X)$ is a strictly angelic space, it is remained to show that for every $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \bar{B}$ there exists a sequence in B converging to \mathbf{x} .

For every $n \in \mathbb{N}$, set $C_n := \pi_n(\bar{B})$. Then $\{C_n\}_{n \in \mathbb{N}}$ is a null sequence of compact subsets of X by Fact 2.21(3).

For every $n \in \mathbb{N}$, set $B_n := p_n(B)$. As B is relatively compact in $\mathfrak{F}_0(X)$, the set B_n is relatively compact in X^n and $p_n(\mathbf{x}) \in \overline{B_n}$. Since B_n is relatively countably compact and X^n is strictly angelic [41], for every $n \in \mathbb{N}$ there is a sequence $\{x_{n,k}\}_{k \in \mathbb{N}}$ in B_n converging to $p_n(\mathbf{x})$. Take arbitrarily a sequence $\{\mathbf{x}_{n,k}\}_{k \in \mathbb{N}}$ in B such that $p_n(\mathbf{x}_{n,k}) = x_{n,k}$. Set $S := \{\mathbf{x}_{n,k}\}_{n,k \in \mathbb{N}}$ and $Z := \text{cl}_{\mathfrak{F}_0(X)}(S)$. Then $S \subseteq B$ and Z is a separable compact subset of \bar{B} . As we proved above, Z is first countable. Thus, to show the existence of a sequence in B converging to \mathbf{x} it is enough to prove that $\mathbf{x} \in Z$.

Fix arbitrarily $U \in \mathcal{N}(X)$. Take a symmetric $V \in \mathcal{N}(X)$ such that $V + V \subseteq U$. Choose $m \in \mathbb{N}$ such that $C_n \subset V$ for every $n > m$. Then $\pi_n(\mathbf{x})$ and $\pi_n(\mathbf{x}_{l,k})$ belong to V for every $n > m$ and each $l, k \in \mathbb{N}$. So $\pi_n(\mathbf{x}_{l,k}) - \pi_n(\mathbf{x}) \in V + V \subseteq U$ for every $n > m$ and each $l, k \in \mathbb{N}$. Take $k_0 \in \mathbb{N}$ such that $p_m(\mathbf{x}_{m,k_0}) - p_m(\mathbf{x}) \in U^n$. Clearly, $\mathbf{x}_{m,k_0} - \mathbf{x} \in U^n \cap c_0(X)$. Thus $\mathbf{x} \in Z$.

(iii) Assume that X is a \check{S} -space and let K be a compact subset of $\mathfrak{F}_0(X)$. We have to prove that K is sequentially compact.

For every $n \in \mathbb{N}$, set $K_n := \pi_n(K)$. Then K_n is compact in X . By assumption, K_n is sequentially compact for every $n \in \mathbb{N}$. Hence $K' := \prod_{n \in \mathbb{N}} K_n$ is sequentially compact by [31, 3.10.35]. Note that K' is a compact subset of $\mathfrak{F}_0(X)$ by Fact 2.21(3), and $K \subseteq K'$. Thus K is sequentially compact by [31, 3.10.33]. Therefore $\mathfrak{F}_0(X)$ is a \check{S} -space. \square

Recall that a topological group X is said to have a *subgroup topology* if it has a base at the identity consisting subgroups.

Proposition 3.1. *Let X be an Abelian topological group with subgroup topology. Then*

- (i) $\mathfrak{F}_0(X)$ has a subgroup topology.
- (ii) $\mathfrak{F}_0(X)$ embeds into a product of discrete Abelian groups. In particular, $\mathfrak{F}_0(X)$ is nuclear.

Proof. (i) Let \mathcal{B} be a base at zero consisting subgroups. Then $\{U^{\mathbb{N}} \cap c_0(X) : U \in \mathcal{B}\}$ is a base at zero in $\mathfrak{F}_0(X)$ consisting subgroups. Thus $\mathfrak{F}_0(X)$ has a subgroup topology.

(ii) By item (i), $\mathfrak{F}_0(X)$ has a subgroup topology. Hence $\mathfrak{F}_0(X)$ embeds into a product of discrete Abelian groups (see, for example, Proposition 2.2 of [4]). By [6, 7.5, 7.6 and 7.10], $\mathfrak{F}_0(X)$ is nuclear. \square

Let X be an Abelian topological group. For a subset A of X , we denote by A^\triangleright the set $\{\chi \in X^\wedge : \chi(A) \subseteq \mathbb{T}_+\}$. A subset A of X is called *quasi-convex* if for every $x \in X \setminus A$ there exists $\chi \in A^\triangleright$ which satisfies $\chi(x) \notin \mathbb{T}_+$. An Abelian topological group is called *locally quasi-convex* if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets.

The next proposition is an immediate corollary of Fact 1.6(i) and [11, Theorem 14].

Proposition 3.2. *If X is a locally quasi-convex Abelian group which admits a coarser metrizable group topology, then $\mathfrak{F}_0(X)^+$ is strictly angelic.*

Denote by **TVS** (respectively, **LCS**) the subcategory of **TAG** consisting of all real topological vector space (TVS for short) (respectively, real locally convex spaces, LCS for short). The following proposition shows that **TVS** and **LCS** are \mathfrak{F}_0 -invariant. For a subset A of a TVS L and an $\alpha \in \mathbb{R}$ we set $\alpha A := \{\alpha a \in L : a \in A\}$.

Proposition 3.3. (i) *If L is a real (respectively, complex) TVS, then $\mathfrak{F}_0(L)$ is a real (respectively, complex) TVS as well.*

(ii) *If L is a real LCS, then $\mathfrak{F}_0(L)$ is also a real LCS.*

Proof. (i) Set $\mathfrak{U} = \{\mathbf{U} : U \in \mathcal{N}(L)\}$, where $\mathbf{U} := U^{\mathbb{N}} \cap c_0(L)$. We have to check the next three conditions (see [49, §15.2]):

- (a) For each $\mathbf{U} \in \mathfrak{U}$ there is a $\mathbf{V} \in \mathfrak{U}$ with $\mathbf{V} + \mathbf{V} \subseteq \mathbf{U}$.
- (b) For each $\mathbf{U} \in \mathfrak{U}$ there is a $\mathbf{V} \in \mathfrak{U}$ for which $\alpha\mathbf{V} \subseteq \mathbf{U}$ for all α with $|\alpha| \leq 1$.
- (c) For each $\mathbf{U} \in \mathfrak{U}$ and each $\mathbf{x} \in \mathfrak{F}_0(L)$ there is a $k \in \mathbb{N}$ for which $\mathbf{x} \in k\mathbf{U}$.

Since L is a TVS, take $V \in \mathcal{N}(L)$ such that $V + V \subseteq U$ and $\alpha V \subseteq U$ for all α with $|\alpha| \leq 1$. Clearly, \mathbf{V} satisfies (a) and (b). Let us check (c). For $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \mathfrak{F}_0(L)$ choose $m \in \mathbb{N}$ such that $x_n \in V$ for every $n > m$. Since L is a TVS, take $k \in \mathbb{N}$ such that $x_1, \dots, x_m \in kV \subseteq kU$. Note that $\frac{1}{k}V \subseteq U$. So, for every $n > m$ we have $x_n = k \cdot (\frac{1}{k}x_n) \in kU$. Now it is clear that $\mathbf{x} \in k\mathbf{U}$.

(ii) By item (i), $\mathfrak{F}_0(L)$ is a real TVS. Since L is locally quasi-convex group by [6, 2.4], the group $\mathfrak{F}_0(L)$ is also locally quasi-convex by Fact 1.6(i). Applying [6, 2.4] once again we obtain that $\mathfrak{F}_0(L)$ is a real locally convex space. \square

Proposition 3.3, Fact 1.6(i) and [11, Corollary 16] immediately imply:

Corollary 3.4. *If L is a real complete LCS, then $\mathfrak{F}_0(L)^+$ is an (E) -space.*

Remark 3.5. Under Martin's Axiom, van Douwen [30] announced the existence of countably compact topological group X for which X^2 is not countably compact. Under the weaker assumption, Hart and van Mill [43] constructed a topological group X whose square is not countably compact. Their result was generalized by Tomita [65]. Shibakov showed that, under CH, the square of a countable Fréchet-Urysohn Abelian group can be not Fréchet-Urysohn [60] and even not sequential (see [59]). Further results and questions in this direction see [50, 59]. So, under additional hypothesis and taking into account that \mathfrak{F}_0 contains X^2 as a closed subgroup, the functor \mathfrak{F}_0 does not preserve countable compactness, sequentiality and Fréchet-Urysohnness. We do not know whether there exists an angelic Abelian topological group whose square is not angelic. Note only that (under CH) [10] provides an example of a nonangelic product of two compact Hausdorff angelic spaces.

4. RESPECTED PROPERTIES FOR TOPOLOGICAL GROUPS

For a Tychonoff topological space X , we denote by $\mathcal{S}(X)$ ($\mathcal{C}(X)$, $\mathcal{SC}(X)$, $\mathcal{CC}(X)$, $\mathcal{PC}(X)$ and $\mathcal{FB}(X)$, respectively) the set of all converging sequences in X with the limit point (respectively, the set of all compact, sequentially compact, countably compact, pseudocompact and functionally bounded in X subsets of X). Note that continuous images and finite disjoint unions of compact sets (respectively, sequentially compact, countably compact, pseudocompact and functionally bounded in X sets) are compact sets (sequentially compact, countably compact, pseudocompact and functionally bounded in X sets, respectively) (see [31]). The following diagram of inclusions is well-known (see [31, §3.10]):

$$(4.1) \quad \begin{array}{ccccc} & & \mathcal{C}(X) & & \\ & \nearrow & & \searrow & \\ \mathcal{S}(X) & & & & \mathcal{CC}(X) \longrightarrow \mathcal{PC}(X) \longrightarrow \mathcal{FB}(X). \\ & \searrow & & \nearrow & \\ & & \mathcal{SC}(X) & & \end{array}$$

In general, if \mathcal{P} is a topological property and X is a topological space, we denote by $\mathcal{P}(X)$ the set of all subspaces of X with \mathcal{P} . In what follows we consider the following families of topological properties

$$\mathfrak{F}_0 := \{\mathcal{S}, \mathcal{C}, \mathcal{SC}, \mathcal{CC}, \mathcal{PC}\} \quad \text{and} \quad \mathfrak{F} := \mathfrak{F}_0 \cup \mathcal{FB}.$$

Following [57], we define:

Definition 4.1. ([57]) *If \mathcal{P} denotes a topological property, then we say that a MAP group X respects \mathcal{P} if $\mathcal{P}(X) = \mathcal{P}(X^+)$.*

So, for example, X respects sequentiality or compactness if $\mathcal{S}(X) = \mathcal{S}(X^+)$ or $\mathcal{C}(X) = \mathcal{C}(X^+)$, respectively.

Diagram 4.1 immediately implies the next simple necessary condition when a MAP group respects one of the properties from \mathfrak{P} :

Proposition 4.2. *Let X be a MAP topological group. If X respects one of the properties from \mathfrak{P} , then every convergent sequence in X^+ with the limit point is functionally bounded in X . In other words, if X^+ has a convergent sequence with the limit point which is not functionally bounded in X , then X does not respect any property from \mathfrak{P} .*

This proposition and Diagram 4.1 motivate the following definition.

Definition 4.3. *A MAP topological group X is said to have the sequentially bounded property (SB-property for short) if every functionally bounded sequence in X^+ is also functionally bounded in X .*

Proposition 4.4. *Let X be a MAP topological group. If X has the SB-property, then every functionally bounded subset in X^+ is precompact in X .*

Proof. Let A be a functionally bounded subset in X^+ . Suppose for a contradiction that A is not left (right) precompact in X . By Proposition 2.3, there exists an infinite sequence $\{u_n\}_{n \in \mathbb{N}}$ in A which is left (right) uniformly discrete in X . Now Lemma 2.5 implies that $\{u_n\}$ is not functionally bounded in X . But, as a subset of A , the sequence $\{u_n\}$ is functionally bounded in X^+ . This contradicts the SB-property. Thus A is left and right precompact, and hence it is a precompact subset of X . \square

Corollary 4.5. *If a MAP topological group X has the \mathbf{sp} -property, then the following are equivalent:*

- (i) X has the SB-property.
- (ii) Every functionally bounded subset of X^+ is precompact in X .

Proof. (i) \Rightarrow (ii) follows from Proposition 4.4.

(ii) \Rightarrow (i) Let $A = \{x_n\}_{n \in \mathbb{N}}$ be a functionally bounded sequence in X^+ . So A is precompact in X . As X has the \mathbf{sp} -property, the closure $B := \text{cl}_X(A)$ is compact in X . So B is functionally bounded in X . Thus A is also functionally bounded in X . Therefore X has the SB-property. \square

Now we obtain a sufficient condition when a *complete* MAP group X respects one of the properties from \mathfrak{P} . Recall that a topological space Y is called a μ -space if every functionally bounded subset of Y is relatively compact.

Proposition 4.6. *Let X be a complete MAP topological group. If X has the SB-property, then*

- (i) X respects each property from \mathfrak{P} ;
- (ii) X^+ is a μ -space.

Proof. Let $\mathcal{P} \in \mathfrak{P}$ be arbitrary. Denote by $id : X \rightarrow X^+$ the identity map. Let A be a subset of X^+ having the property \mathcal{P} . Then A is functionally bounded in X^+ by Diagram 4.1. By Lemma 2.4, the closure $B := \text{cl}_{X^+}(A)$ is also functionally bounded in X^+ . Now Proposition 4.4 implies that B is a closed precompact subset of X . So B is a compact subset of X by Fact 2.7. Hence B is compact in X^+ . Since the restriction $id|_B$ of id onto B is a homeomorphism we obtain that A also has \mathcal{P} . Thus X respects \mathcal{P} that proves item (i). Now the partial case in this proof when $\mathcal{P} = \mathcal{FB}$ shows that X^+ is a μ -space. \square

Taking into account [47, Theorem 3.2] the next theorem generalizes the equivalence of (a)-(d) in Theorem 3.3 of [47]:

Theorem 4.7. *Let X be a complete MAP topological group. Then the following are equivalent:*

- (i) X respects compactness and X^+ is a μ -space.

- (ii) X respects countable compactness and X^+ is a μ -space.
- (iii) X respects pseudocompactness and X^+ is a μ -space.
- (iv) X respects functional boundedness and X^+ is a μ -space.
- (v) X has the SB -property.

Hence, if (i)-(v) hold, then X respects sequentiality and sequential compactness as well.

Proof. (i)-(iv) \Rightarrow (v) Let A be a functionally bounded sequence in X^+ . As X^+ is a μ -space, the set $B := \text{cl}_{X^+}(A)$ is compact in X^+ . By hypothesis on X and Diagram 4.1, B is functionally bounded in X . Thus A is a functionally bounded subset of X (see Lemma 2.4). Therefore X has the SB -property.

(v) implies (i)-(iv) and the last assertion by Proposition 4.6. \square

For the properties from \mathfrak{P}_0 we complete Proposition 4.6 by the following:

Proposition 4.8. *Let X be a MAP topological group and $\mathcal{P} \in \mathfrak{P}_0$. If the completion \overline{X} of X respects \mathcal{P} , then X also respects \mathcal{P} .*

Proof. Denote by $i : X \rightarrow \overline{X}$ the natural embedding. Since $bX = b\overline{X}$ by Lemma 2.13, the identity map $i^+ : X^+ \rightarrow \overline{X}^+, i^+(x) = x$, is also an embedding. Let $A \in \mathcal{P}(X^+)$. Then $A = i^+(A) \in \mathcal{P}(\overline{X}^+)$. Since \overline{X} respects \mathcal{P} , we have $A \in \mathcal{P}(\overline{X})$. Hence $A \in \mathcal{P}(X)$ as well. Thus X respects \mathcal{P} . \square

Assume that a subcategory \mathbf{A} of \mathbf{MAP} is closed under taking of completions. Then Proposition 4.8 shows that $\mathcal{P} \in \mathfrak{P}_0$ is a respected property in \mathbf{A} if and only if every complete $X \in \mathbf{A}$ respects \mathcal{P} .

The next proposition is an analogue of Proposition 2.1 in [55]:

Proposition 4.9. *Let $\{X_i\}_{i \in I}$ be a non-empty family of MAP Abelian groups and $X := \prod_{i \in I} X_i$. Then*

- (i) ([55]) X respects compactness if and only if X_i respects compactness for every $i \in I$.
- (ii) X has the Schur property if and only if X_i has the Schur property for every $i \in I$.
- (iii) X respects functional boundedness if and only if X_i respects functional boundedness for every $i \in I$.
- (iv) X has the SB -property if and only if X_i has the SB -property for every $i \in I$.
- (v) If X respects pseudocompactness (respectively, countable compactness or sequential compactness), then X_i respects pseudocompactness (respectively, countable compactness or sequential compactness) for every $i \in I$.

Proof. Denote by π_i the projection from X onto X_i . Define $\nu_j : X_j \rightarrow X$ as follows:

$$\nu_j(x_j) := (x_i), \text{ where } x_i = x_j \text{ if } i = j, \text{ and } x_i = 0 \text{ otherwise.}$$

(ii) Let X have the Schur property. Fix arbitrarily an index $j \in I$. We have to show that X_j has the Schur property. Let a sequence $\{x_j^n\}_{n \in \omega}$ in X_j converge to zero in X_j^+ . For every $n \in \mathbb{N}$ we set $\mathbf{x}_n = \nu_j(x_j^n)$. Then for each $\chi \in \widehat{X}$, we have $(\chi, \mathbf{x}_n) = (\chi|_{X_j}, x_j^n) \rightarrow 1$. Thus $\mathbf{x}_n \rightarrow 0$ in $\sigma(X, \widehat{X})$. Since X has the Schur property, $\mathbf{x}_n \rightarrow 0$ in the product topology on X . Hence $x_j^n = \pi_j(\mathbf{x}_n) \rightarrow 0$ in X_j . Thus X_j has the Schur property.

Conversely, let X_i have the Schur property for every $i \in I$, and let $\{\mathbf{x}_n\}_{n \in \omega}$ be a null sequence in X^+ . By Fact 2.14, $\pi_i(\mathbf{x}_n) \rightarrow 0$ in X_i^+ for every $i \in I$. Since X_i has the Schur property, $\pi_i(\mathbf{x}_n) \rightarrow 0$ in X_i for every $i \in I$. Now, for every $U \in \mathcal{N}(X)$ of the form

$$U = U_{i_1} \times \cdots \times U_{i_t} \times \prod_{i \in I \setminus \{i_1, \dots, i_t\}} X_i, \text{ where } U_{i_k} \in \mathcal{N}(X_{i_k}) \text{ and } 1 \leq k \leq t,$$

choose $m \in \mathbb{N}$ such that $\pi_{i_k}(\mathbf{x}_n) \in U_{i_k}$ for every $n \geq m$ and each $1 \leq k \leq t$. Then $\mathbf{x}_n \in U$ for every $n \geq m$. This means that $\mathbf{x}_n \rightarrow 0$ in X . Thus X has the Schur property.

(iii),(iv) Let X respect functional boundedness (respectively, X have the SB -property). Fix arbitrarily an index $j \in I$. We have to show that X_j respects functional boundedness (respectively, X_j has the SB -property). Let A_j be a functionally bounded subset (respectively, sequence) in X_j^+ . Set $A := \nu_j(A_j)$. Then

A is functionally bounded in $X_j^+ \times \left(\prod_{i \in I, i \neq j} X_i\right)^+ = X^+$ by Lemma 2.4 and Fact 2.14. By assumption, A is functionally bounded in X . Hence $\pi_j(A) = A_j$ is functionally bounded in X_j . Thus X_j respects functional boundedness (respectively, X_j has the SB -property).

Conversely, let X_i respect functional boundedness (respectively, X_i have the SB -property) for every $i \in I$, and let A be a functionally bounded subset (respectively, sequence) in X^+ . By Lemma 2.4 and Fact 2.14, the set $\pi_i(A)$ is functionally bounded in X_i^+ , and hence in X_i , for every $i \in I$. Now Theorem 2.2 of [64] implies that the set $E := \prod_{i \in I} \pi_i(A)$ is functionally bounded in X . Since $A \subseteq E$, A is also functionally bounded in X by Lemma 2.4. Thus X respects functional boundedness (respectively, X has the SB -property).

(v) We consider only the pseudocompact case. Fix arbitrarily an index $j \in I$. We have to show that X_j respects pseudocompactness. Taking into account that each topological group is Tychonoff, we note that the image of a pseudocompact subset under a continuous homomorphism is pseudocompact [31, 3.10.24]. Now let A be a pseudocompact subset in X_j^+ . Then $\nu_j(A)$ is a pseudocompact subset of $X^+ = X_j^+ \times \left(\prod_{i \in I, i \neq j} X_i\right)^+$. Since X respects pseudocompactness, A is pseudocompact in X . Now $\pi_j(\nu_j(A)) = A$ is pseudocompact in X_j . \square

In the next proposition we consider heredity of respected properties. Note that, for the Schur property, item (i) was noticed in [51, §2], and, for respect compactness, this item is folklore (for dually embedded subgroups it is proved in [55, 2.4]):

Proposition 4.10. *Let H be a subgroup of a MAP Abelian group X and $\mathcal{P} \in \mathfrak{P}_0$.*

- (i) *If X respects \mathcal{P} , then H respects \mathcal{P} as well.*
- (ii) *If H is open and respects \mathcal{P} , then X respects \mathcal{P} .*

Proof. (i) Let $K \in \mathcal{P}(H^+)$. Denote by $i : H^+ \rightarrow X^+$ the identity map. By Lemma 2.11(3), i is continuous. So $K = i(K) \in \mathcal{P}(X^+)$ (note that Hausdorff topological groups are Tychonoff). Hence $K \in \mathcal{P}(X)$. Thus $K \in \mathcal{P}(H)$. Therefore H respects \mathcal{P} .

(ii) Let $K \in \mathcal{P}(X^+)$. Since K is also functionally bounded in X^+ , K is contained in a finite union of cosets of H by Lemma 2.12. Since H is open, Fact 2.9 and Lemma 2.11(4) imply that H is a closed subgroup of X^+ . Hence, without loss of generality, we can assume that $K \subseteq H^+$ and $K \in \mathcal{P}(H^+)$. As H respects \mathcal{P} , $K \in \mathcal{P}(H)$, and hence $K \in \mathcal{P}(X)$. Thus X respects \mathcal{P} . \square

For functional boundedness and the SB -property the situation is more complicated. Recall that a subspace E of a topological space Y is called C -embedded (respectively, C^* -embedded) in Y if every continuous function (respectively, every bounded continuous function) on E can be extended to a continuous function (respectively, to a bounded continuous function) on Y .

Proposition 4.11. *Let H be a subgroup of a MAP Abelian group X .*

- (i) *If X respects functional boundedness and H is C -embedded in X , then H also respects functional boundedness.*
- (ii) *Let H be open and respect functional boundedness. If H^+ is C -embedded in X^+ , then X respects functional boundedness.*
- (iii) *If X has the SB -property and H is C -embedded in X , then H also has the SB -property.*
- (iv) *Let H be open and have the SB -property. If H^+ is C -embedded in X^+ , then X has the SB -property.*

Proof. (i),(iii) Let A be a functionally bounded subset (respectively, sequence) in H^+ . As $\sigma(X, \widehat{X})|_H \leq \sigma(H, \widehat{H})$ by Lemma 2.11, A is also functionally bounded in X^+ . Hence, by assumption, A is functionally bounded in X . Since H is C -embedded, A is functionally bounded in H . Thus H respects functional boundedness (respectively, H has the SB -property).

(ii),(iv) Let A be a functionally bounded subset (respectively, sequence) in X^+ . By Lemma 2.12, A is contained in a finite union of cosets of H . Hence, by Lemma 2.4(1), without loss of generality we can assume that $A \subseteq H$.

Fact 2.9 and Lemma 2.11 imply that H^+ is a closed subgroup of X^+ . Since H^+ is C -embedded in X^+ , A is functionally bounded in H^+ . So, by assumption, A is functionally bounded in H . Thus A is functionally bounded in X . Therefore, X respects functional boundedness (respectively, X has the SB -property). \square

We do not know whether it is possible to weaken conditions on H and H^+ in Proposition 4.11. Note that by Lemma 2.11(5), a subgroup H of a MAP Abelian group X is dually closed and dually embedded if and only if H^+ is a closed subgroup of X^+ . So the next general question is of interest:

Problem 4.12. *Let X be a MAP Abelian group. Characterize those dually closed and dually embedded subgroups H of X such that H^+ is C -embedded in X^+ .*

If G is a LCA group, it is well-known that every closed subgroup S of G is dually closed and dually embedded in G . Moreover, S^+ is automatically C -embedded in G^+ by Theorem 5.6 of [14]. It is known (see [6, 8.3 and 8.6]) that every closed subgroup of a nuclear group is dually closed and dually embedded. This justifies the next question which is a partial case of [14, Problem 7.5]:

Problem 4.13. *Let H be a closed subgroup of a nuclear group X . Is H^+ C -embedded in X^+ ?*

It is also interesting to consider Problem 4.12 for Schwartz groups. In the next question we reformulate and extend Problem 4.12:

Problem 4.14. *Characterize C -embedded and C^* -embedded closed subgroups of precompact (Abelian) groups.*

5. PROOF OF THEOREM 1.10

We prove Theorem 1.10 by a reduction of the general case to several simpler ones using the structure theory of LCA groups and the results of the previous section. We need some lemmas.

It is well-known that the space c_0 does not have the Schur property. Below we generalize this fact.

Lemma 5.1. *The space $c_0 = \mathfrak{F}_0(\mathbb{R})$ does not respect any property from \mathfrak{P} .*

Proof. By Proposition 4.2 it is enough to find a convergent sequence in X^+ which is not functionally bounded in X . Set $e_0 := 0 \in c_0$ and $e_n := \nu_n(1)$ for every $n \in \mathbb{N}$. We claim that $e_n \rightarrow e_0$ in c_0^+ . Indeed, for every $\mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \ell_1 = c_0^\wedge$, we have $(\mathbf{s}, e_n) = s_n \rightarrow 0$. Thus $e_n \rightarrow e_0$ in c_0^+ .

On the other hand, the set $K := \{e_n\}_{n \in \omega}$ is not functionally bounded in c_0 . Indeed, since $\|e_n - e_m\| \geq 1$ for all distinct $n, m \in \omega$, the set K is uniformly discrete in c_0 . By Lemma 2.5, the set K is not functionally bounded in c_0 . \square

Next proposition generalizes Lemma 5.1.

Proposition 5.2. *If L is a nontrivial real LCS, then $\mathfrak{F}_0(L)$ does not respect any property from \mathfrak{P} .*

Proof. Since $L \neq \{0\}$ we can represent L in the form $L = \mathbb{R} \oplus L_0$, where L_0 is a closed subspace of L [49, §20.5(5)]. So $\mathfrak{F}_0(L) \cong c_0 \oplus \mathfrak{F}_0(L_0)$ by Fact 2.21. Now the assertion follows from Proposition 4.9 and Lemma 5.1. \square

The connected component of a topological group X we denote by $C(X)$.

Lemma 5.3. *Let X be a LCA group containing an open compact subgroup. If $C(X) \neq \{0\}$, then $\mathfrak{F}_0(X)$ does not respect any property from \mathfrak{P} .*

Proof. We use Proposition 4.2. Take arbitrarily a non-zero $x \in C(X)$. Set $\mathbf{x}_0 := 0 \in \mathfrak{F}_0(X)$ and $\mathbf{x}_n := \nu_n(x)$ for every $n \in \mathbb{N}$. If a symmetric $U \in \mathcal{N}(X)$ is such that $x \notin U$, then $K := \{\mathbf{x}_n\}_{n \in \omega}$ is $(U^\mathbb{N} \cap c_0(X))$ -separated. Thus K is not functionally bounded in $\mathfrak{F}_0(X)$ by Lemma 2.5. So to prove the lemma it is enough to show that $\mathbf{x}_n \rightarrow \mathbf{x}_0$ in $\widehat{\mathfrak{F}_0(X)}$.

Let $\mathbf{g} = (g_n)_{n \in \mathbb{N}} \in \widehat{\mathfrak{F}_0(X)}$. By Fact 2.22(2), there exists an open subgroup H of X and a natural number m such that $g_n \in H^\perp$ for every $n \geq m$. Noting that $C(X) \subseteq H$, we obtain that $(\mathbf{g}, \mathbf{x}_n) = (g_n, x) = 1$ for every $n \geq m$. Thus $\mathbf{x}_n \rightarrow \mathbf{x}_0$ in $\widehat{\mathfrak{F}_0(X)}$. \square

In spite of the next two lemmas follows from [9, Theorem] and Proposition 3.1(ii), we give here their independent proofs.

Lemma 5.4. *Let X be a totally disconnected LCA group. If E is a functionally bounded subset in $\mathfrak{F}_0(X)^+$, then E is functionally bounded also in $\mathfrak{F}_0(X)$. Moreover, $\text{cl}_{\mathfrak{F}_0(X)}(E)$ is a compact subset of $\mathfrak{F}_0(X)$ and $\mathfrak{F}_0(X)^+$.*

Proof. We show first that the sequence $\{\pi_n(E)\}_{n \in \mathbb{N}}$ is a null-sequence in X . Suppose for a contradiction that $\{\pi_n(E)\}_{n \in \mathbb{N}}$ is not a null-sequence. Then, by Lemma 2.20, E has a uniformly discrete sequence $\{\mathbf{b}_n\}_{n \in \mathbb{N}}$. Since X is totally disconnected it has a subgroup topology [48, 7.7]. Hence $\mathfrak{F}_0(X)$ has a subgroup topology by Proposition 3.1(i). So there exists an open subgroup H of X such that $\{\mathbf{b}_n\}$ is $(H^{\mathbb{N}} \cap c_0(X))$ -separated. Thus $\{\mathbf{b}_n\}$, and hence E , is not contained in a finite union of cosets of $H^{\mathbb{N}} \cap c_0(X)$. As $H^{\mathbb{N}} \cap c_0(X)$ is an open subgroup of $\mathfrak{F}_0(X)$ by Fact 2.21(2), Lemma 2.12 implies that E is not functionally bounded in $\mathfrak{F}_0(X)^+$, a contradiction.

Since the set $\pi_n(E)$ is functionally bounded in X^+ for every $n \in \mathbb{N}$ by Lemma 2.4, Theorem 4.2 of [67] implies that $\pi_n(E)$ is also functionally bounded in X and $K_n := \text{cl}_X(\pi_n(E))$ is compact in X . Now Lemma 2.19 implies that $\{K_n\}_{n \in \mathbb{N}}$ is a null-sequence of compact subsets in X . Set $K := \prod_{n \in \mathbb{N}} K_n$. Fact 2.21(3) yields that K is a compact subset of $\mathfrak{F}_0(X)$. Clearly, $\text{cl}_{\mathfrak{F}_0(X)}(E) \subseteq K$. Hence $\text{cl}_{\mathfrak{F}_0(X)}(E)$ is compact in $\mathfrak{F}_0(X)$. Thus E is functionally bounded in $\mathfrak{F}_0(X)$ by Lemma 2.4. \square

Lemma 5.5. *Let X be a totally disconnected LCA group. If E is a pseudocompact (respectively, countably compact or sequentially compact) subset of $\mathfrak{F}_0(X)^+$, then E is pseudocompact (respectively, countably compact or sequentially compact) also in $\mathfrak{F}_0(X)$.*

Proof. Since E is pseudocompact (respectively, countably compact or sequentially compact) in $\mathfrak{F}_0(X)^+$, then it is also functionally bounded in $\mathfrak{F}_0(X)^+$. Hence $K := \text{cl}_{\mathfrak{F}_0(X)}(E)$ is compact in $\mathfrak{F}_0(X)$ by Lemma 5.4. So the topologies of $\mathfrak{F}_0(X)$ and $\mathfrak{F}_0(X)^+$ coincide on K . In particular, $\mathbf{u}_0|_E = \mathbf{u}_0^+|_E$. Thus E is pseudocompact (respectively, countably compact or sequentially compact) in $\mathfrak{F}_0(X)$. \square

The class of nuclear groups was introduced by Banaszczyk in [6, Chapter 3]. The concept of a Schwartz topological Abelian group appeared in [3]. This notion generalizes the well-known notion of a Schwartz locally convex space. Recall that an Abelian Hausdorff group X is called a *Schwartz group* if for every $U \in \mathcal{N}(X)$ there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite subsets of X such that the intersection $\bigcap_{n \in \mathbb{N}} ((1/n)U + F_n)$ is a neighborhood of 0, where $(1/n)U := \{x \in X : kx \in U \forall 1 \leq k \leq n\}$. All nuclear groups as well as the Pontryagin dual groups of metrizable Abelian groups are Schwartz groups (see [3]).

Now we are in position to prove Theorem 1.10.

Proof of Theorem 1.10. (i) \Rightarrow (ii). Let X be totally disconnected. Then X has a subgroup topology by [48, 7.7]. Hence $\mathfrak{F}_0(X)$ also has a subgroup topology by Proposition 3.1(i). Thus $\mathfrak{F}_0(X)$ embeds into a direct product of discrete Abelian groups (see, for example, Proposition 2.2 in [4]).

(ii) \Rightarrow (iii). Every LCA group is nuclear [6, 7.10]. Thus $\mathfrak{F}_0(X)$ is nuclear by [6, 7.5 and 7.6].

(iii) \Rightarrow (iv). Every nuclear group is a Schwartz groups by [3].

(iv) \Rightarrow (v) follows from Theorem 4.4 of [2] since $\mathfrak{F}_0(X)$ is locally quasi-convex by Fact 1.6(i).

(v) \Rightarrow (vi) is clear (see also [11]).

(vi) \Rightarrow (i), (vii) \Rightarrow (i), (viii) \Rightarrow (i), (ix) \Rightarrow (i), (x) \Rightarrow (i) Let $X \cong \mathbb{R}^n \times X_0$, where $n \geq 0$ and X_0 has an open compact subgroup K [48, 24.30].

By Fact 2.22, $\mathfrak{F}_0(X) \cong c_0^n \times \mathfrak{F}_0(X_0)$. Since c_0 does not respect sequentiality (respectively, countable compactness, sequential compactness, pseudocompactness and functional boundedness) by Lemma 5.1, we obtain that $n = 0$ by Proposition 4.9. Hence $X = X_0$ has an open compact subgroup K . Now Lemma 5.3 implies that $C(X)$ is trivial. Thus X is totally disconnected.

The implications (i) \Rightarrow (vii), (i) \Rightarrow (viii), (i) \Rightarrow (ix) and (i) \Rightarrow (x) and the last assertion of the theorem follow from Lemmas 5.4 and 5.5. \square

Recall [26] that a Hausdorff topological group X is called *locally q -minimal* if there exists a neighborhood V of the identity e_X such that whenever H is a Hausdorff group and $p : X \rightarrow H$ is a continuous surjective

homomorphism such that $p(V)$ is a neighborhood of e_H , then p is open. The next corollary of Theorem 1.10 and [26, Theorem 1.3] was pointed out to the author by D. Dikranjan:

Corollary 5.6. *If X is a totally disconnected non-discrete LCA group, then $\mathfrak{F}_0(X)$ is not locally q -minimal.*

Proof. By Theorem 1.10, the group $\mathfrak{F}_0(X)$ embeds onto a subgroup H of a product of locally compact groups. Since $\mathfrak{F}_0(X)$ is complete by Fact 1.6(i), H is closed. Fact 1.6(iii) implies that H is not locally compact. So H , and hence $\mathfrak{F}_0(X)$, is not locally q -minimal by [26, Theorem 1.3]. \square

We do not know whether $\mathfrak{F}_0(X)$ is locally q -minimal for a connected LCA group X .

Remark 5.7. Note that the Glicksberg property has been studied beyond LCA groups by many authors, see [2, 8, 9, 38, 57, 71]. The Schur property in Abelian topological groups was intensively studied by Martín-Peinador and Tarieladze in [51] (see also [36, 46]). Many other results concerning preservation and respectness of topological properties under the Bohr functor see [11]. A complete characterization of Banach spaces which have the Schur property is given in [46].

6. PROOF OF THEOREM 1.11

Let X be a metrizable Abelian group and ρ be an invariant metric for X . For every $n \in \mathbb{N}$, set $U_n^X = U_n := \{x \in X : \rho(x, 0) < \frac{1}{n}\}$. Note that the equality

$$d(\mathbf{x}, \mathbf{y}) := \sup_{n \in \mathbb{N}} \rho(x_n, y_n), \quad \forall \mathbf{x} = (x_n), \mathbf{y} = (y_n) \in c_0(X),$$

defines an invariant metric for $\mathfrak{F}_0(X)$ [23]. Analogously, for every $m \in \mathbb{N}$, we consider X^m with the metric

$$d_m((x_1, \dots, x_m), (y_1, \dots, y_m)) := \max\{\rho(x_1, y_1), \dots, \rho(x_m, y_m)\}.$$

So $U_n^{X^m} = U_n^m$ for every $n, m \in \mathbb{N}$.

Recall that a subset A of X is called ε -dense in X if for every $x \in X$ there exists $a \in A$ such that $\rho(x, a) < \varepsilon$. For every $n, m \in \mathbb{N}$, we denote by $B_X(U_m, n)$ the set of all $z \in X$ such that the set $F_n(z) := \{-nz, \dots, -z, 0, z, \dots, nz\}$ is $\frac{1}{m}$ -dense in X .

Lemma 6.1. *Let X be a compact metrizable Abelian group with an invariant metric ρ . Then, for every $n, m \in \mathbb{N}$, the set $B_X(U_m, n)$ is open in X .*

Proof. If $B_X(U_m, n) = \emptyset$, it is open. Assume now that $B_X(U_m, n)$ is not empty. For every $x, z \in X$, we define

$$f(x, z) := \min\{\rho(x, kz) : k \in \{-n, \dots, -1, 0, 1, \dots, n\}\} = \rho(x, F_n(z)).$$

Then $f(x, z)$ is continuous on $X \times X$. Clearly, $F_n(z)$ is $\frac{1}{m}$ -dense in X if and only if $f(x, z) < \frac{1}{m}$ for every $x \in X$.

Fix $z_0 \in B_X(U_m, n)$. Since X is compact, we can set $\varepsilon_0 := \max\{f(x, z_0) : x \in X\}$. Then $\varepsilon_0 < \frac{1}{m}$. As $f(x, z)$ is also uniformly continuous, there is $\delta \in \mathbb{N}$ such that

$$|f(x', z') - f(x, z)| < \frac{1}{2} \left(\frac{1}{m} - \varepsilon_0 \right),$$

for every $(x', z'), (x, z) \in X \times X$ with $(x', z') - (x, z) \in U_\delta \times U_\delta$. In particular, for each $z \in z_0 + U_\delta$, we have

$$|f(x, z) - f(x, z_0)| < \frac{1}{2} \left(\frac{1}{m} - \varepsilon_0 \right), \quad \forall x \in X.$$

So, for every $z \in z_0 + U_\delta$ and each $x \in X$, we obtain

$$f(x, z) < f(x, z_0) + \frac{1}{2} \left(\frac{1}{m} - \varepsilon_0 \right) \leq \varepsilon_0 + \frac{1}{2} \left(\frac{1}{m} - \varepsilon_0 \right) = \frac{1}{2} \left(\frac{1}{m} + \varepsilon_0 \right) < \frac{1}{m}.$$

This means that $F_n(z)$ is $\frac{1}{m}$ -dense in X for every $z \in z_0 + U_\delta$. Therefore, $B_X(U_m, n)$ is open in X . \square

Let X be a compact *connected* metrizable Abelian group and m_X be the normalized Haar measure on X . Denote by $S(X)$ the set of all elements x of X such that the cyclic subgroup $\langle x \rangle$ generated by x is dense in X . By [48, 25.27], $S(X)$ is measurable and

$$(6.1) \quad m_X(S(X)) = 1.$$

Lemma 6.2. *Let X be a compact connected metrizable Abelian group and let z be an element of $S(X)$. Then*

- (1) $nz \in S(X)$ for every $n \in \mathbb{N}$.
- (2) For every $U \in \mathcal{N}(X)$ there exists $n_U \in \mathbb{N}$ which satisfies the next condition: for each $x \in X$ there is an integer number n with $|n| \leq n_U$ such that $x - nz \in U$.

Proof. (1) Set $X_n := \text{cl}(\langle nz \rangle)$. We have to show that $X_n = X$. Suppose for a contradiction that $X_n \neq X$. Then $\langle q(z) \rangle$ is dense in X/X_n , where $q : X \rightarrow X/X_n$ is the quotient map. Since the group $\langle q(z) \rangle$ is finite, we obtain that the connected group X/X_n is non-trivial and finite, a contradiction.

(2) Take a symmetric $V \in \mathcal{N}(X)$ such that $V + V \subseteq U$. Since X is compact, there is a finite subset $\{y_l\}_{l=1}^m$ of X such that $\{y_l + V\}_{l=1}^m$ is a cover of X . For every $1 \leq l \leq m$, choose $n_l \in \mathbb{Z}$ such that $n_l z \in y_l + V$ and set $n_U := \max\{|n_1|, \dots, |n_m|\}$.

Let $x \in X$ be arbitrary. Take $1 \leq l \leq m$ such that $x \in y_l + V$. Then $x - n_l z \in V - V \subseteq U$ and $|n_l| \leq n_U$. Thus n_U is as desired. \square

Let X be a compact connected metrizable Abelian group. For every $m, n \in \mathbb{N}$, set

$$A_X(U_m, n) := B_X(U_m, n) \cap S(X).$$

Then $A_X(U_m, n) \subseteq A_X(U_m, n+1)$, for every $n, m \in \mathbb{N}$, and Lemma 6.2(2) implies that

$$(6.2) \quad S(X) = \bigcup_{n \in \mathbb{N}} A_X(U_m, n), \quad \forall m \in \mathbb{N}.$$

By Lemma 6.1, $A_X(U_m, n)$ is a measurable subset of X for every $m, n \in \mathbb{N}$.

Proposition 6.3. *Let X be a compact connected metrizable Abelian group. Then the group $\mathfrak{F}_0(X)$ is monothetic.*

Proof. Let ρ be an invariant metric for X . We separate the proof into two steps.

Step 1. Let us show first that for every $m \in \mathbb{N}$ there exists a natural number n_m and a compact subset K_m of the direct product X^m such that the sequences $\{n_m\}$ and $\{K_m\}$ satisfy the following conditions:

- (i) $1 = n_0 < n_1 < n_2 < \dots$;
- (ii) $K_m \subseteq A_{X^m} \left((U_{2^{m n_{m-1}}})^m, n_m \right) = A_{X^m} \left(U_{2^{m n_{m-1}}}^{X^m}, n_m \right)$ for every $m \in \mathbb{N}$;
- (iii) $K_{m+1}|_{X^m} \subseteq K_m$ for every $m \in \mathbb{N}$;
- (iv) $\pi_m(K_m) \subseteq U_{2^{m n_{m-1}}}$ for every $m \in \mathbb{N}$;
- (v) $m_{X^m}(K_m) > 0$ for every $m \in \mathbb{N}$.

We build such sequences $\{n_m\}$ and $\{K_m\}$ by induction. For $m = 1$, by (6.1) and (6.2), we can choose $n_1 > 1$ such that

$$m_X(A_X(U_2, n_1) \cap U_2) > 0.$$

Take arbitrarily a compact subset K_1 of $A_X(U_2, n_1) \cap U_2$ such that $m_X(K_1) > 0$. Clearly, n_1 and K_1 satisfy (i)-(v).

Suppose we have already built $1 = n_0 < n_1 < \dots < n_m$ and K_1, \dots, K_m which satisfy (i)-(v). Since $m_{X^{m+1}}(K_m \times U_{2^{m+1} n_m}) > 0$, (6.1) and (6.2) imply that there is $n_{m+1} > n_m$ such that

$$m_{X^{m+1}}(E_{m+1}) > 0,$$

where $E_{m+1} := A_{X^{m+1}} \left((U_{2^{m+1} n_m})^{m+1}, n_{m+1} \right) \cap (K_m \times U_{2^{m+1} n_m})$. Now take arbitrarily a compact subset K_{m+1} of E_{m+1} such that $m_{X^{m+1}}(K_{m+1}) > 0$. Clearly, n_1, \dots, n_{m+1} and K_1, \dots, K_{m+1} satisfy (i)-(v).

Step 2. For every $m \in \mathbb{N}$, set $K'_m := K_m \times X^{\mathbb{N} \setminus \{1, \dots, m\}}$. Then, by (iii) and (v), $\{K'_m\}$ is a decreasing sequence of non-empty compact subsets of $X^{\mathbb{N}}$. Take arbitrarily $\mathbf{z} = (z_m) \in \bigcap_{m \in \mathbb{N}} K'_m$. Then $z_m \in \pi_m(K_m)$, and hence $\mathbf{z} \in c_0(X)$ by (iv). To prove the proposition it is enough to show that $\langle \mathbf{z} \rangle$ is dense in $\mathfrak{F}_0(X)$.

Let $\varepsilon > 0$ and $\mathbf{x} = (x_n) \in c_0(X)$. Choose $m \in \mathbb{N}$ such that

$$(6.3) \quad \frac{1}{2^m} + \sup_{n > m} \rho(x_n, 0) < \frac{\varepsilon}{2}.$$

Note that $\mathbf{z}_m := (z_1, \dots, z_m) \in K_m$. Hence, by (ii), for $\mathbf{x}_m := (x_1, \dots, x_m)$ there exists an integer number k such that $|k| \leq n_m$ and

$$(6.4) \quad \mathbf{x}_m - k \cdot \mathbf{z}_m = (x_1 - kz_1, \dots, x_m - kz_m) \in (U_{2^m n_{m-1}})^m.$$

It remains to prove that $d(\mathbf{x}, k\mathbf{z}) < \varepsilon$.

For $n = 1, \dots, m$, by (6.3) and (6.4), we have

$$(6.5) \quad \rho(x_n, kz_n) < \frac{1}{2^m n_{m-1}} < \frac{\varepsilon}{2}.$$

Let $n > m$. Since $|k| \leq n_m$, $z_n \in \pi_n(K_n)$ and ρ is invariant, we obtain

$$(6.6) \quad \rho(x_n, kz_n) \leq \rho(x_n, 0) + \rho(0, kz_n) \leq \rho(x_n, 0) + |k| \rho(0, z_n) \stackrel{(i),(iv)}{\leq} \rho(x_n, 0) + n_m \cdot \frac{1}{2^{m+1} \cdot n_m} \stackrel{(6.3)}{<} \frac{\varepsilon}{2}.$$

Now (6.5) and (6.6) imply that $d(\mathbf{x}, k\mathbf{z}) < \varepsilon$. Hence $\langle \mathbf{z} \rangle$ is dense in $\mathfrak{F}_0(X)$. Thus $\mathfrak{F}_0(X)$ is monothetic. \square

Following [17] we say that a topological group X has *property UB* if each real-valued uniformly continuous function on X is bounded. In the next proposition we generalize Example 4.2 in [17].

Proposition 6.4. *Let X be a connected compact Abelian group. Then the groups $(X^{\mathbb{N}}, \mathbf{u})$ and $\mathfrak{F}_0(X)$ have the property UB.*

Proof. Let $f(\mathbf{x})$ be a real-valued uniformly continuous function on $(X^{\mathbb{N}}, \mathbf{u})$ (respectively, on $\mathfrak{F}_0(X)$). Take $U \in \mathcal{N}(X)$ such that whenever $\mathbf{x}, \mathbf{z} \in X^{\mathbb{N}}$ and $\mathbf{x} - \mathbf{z} \in U^{\mathbb{N}}$ (respectively, $\mathbf{x}, \mathbf{z} \in c_0(X)$ and $\mathbf{x} - \mathbf{z} \in U^{\mathbb{N}} \cap c_0(X)$) we have

$$(6.7) \quad |f(\mathbf{x}) - f(\mathbf{z})| < 1.$$

Since X is connected we have

$$X = \bigcup_{n \in \mathbb{N}} (n)U, \text{ where } (n)U := \underbrace{U + \dots + U}_n.$$

As X is also compact, there is a natural number m such that $X = (m)U$.

Fix $\mathbf{x} := (x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$ (respectively, $\mathbf{x} \in \mathfrak{F}_0(X)$). For every $i \in \mathbb{N}$ set $r_i := \min\{l \in \mathbb{N} : x_i \in (l)U\}$. So $r_i \leq m$ for every $i \in \mathbb{N}$ and, if $\mathbf{x} \in \mathfrak{F}_0(X)$, then $r_i = 1$ for all sufficiently large i . Hence for every $i \in \mathbb{N}$ there are nonzero elements $u_{1,i}, \dots, u_{r_i,i} \in U$ such that

$$x_i = u_{1,i} + \dots + u_{r_i,i}.$$

Set $s := \max\{r_i : i \in \mathbb{N}\}$. So $s \leq m$. For every $1 \leq l \leq s$ we set

$$y_i^l := \begin{cases} u_{i,l}, & \text{if } l \leq r_i \\ 0, & \text{if } l > r_i \end{cases},$$

and put $\mathbf{y}_l := (y_i^l)_{i \in \mathbb{N}}$. Clearly, for every $1 \leq l \leq s$ we have $\mathbf{y}_l \in U^{\mathbb{N}}$. In the case $\mathbf{x} \in \mathfrak{F}_0(X)$ we have $\mathbf{y}_1 \in U^{\mathbb{N}} \cap c_0(X)$ and $\mathbf{y}_l \in U^{\mathbb{N}} \cap X^{(\mathbb{N})}$ for every $2 \leq l \leq s$. So

$$(6.8) \quad \mathbf{x} = \mathbf{y}_1 + \dots + \mathbf{y}_s, \text{ where } \mathbf{y}_l \in U^{\mathbb{N}} \left(\text{respectively, } \mathbf{y}_l \in U^{\mathbb{N}} \cap \mathfrak{F}_0(X) \right) \text{ for every } 1 \leq l \leq s.$$

Set $\mathbf{y}_0 := (0)$. Now (6.7) and (6.8) imply

$$|f(\mathbf{x})| \leq |f(\mathbf{y}_0)| + \sum_{i=1}^s |f(\mathbf{y}_0 + \cdots + \mathbf{y}_i) - f(\mathbf{y}_0 + \cdots + \mathbf{y}_{i-1})| \leq |f(\mathbf{y}_0)| + m.$$

Thus f is bounded. □

We do not know whether every real-valued uniformly continuous function on $\mathfrak{F}_0(X)$ can be extended to a real-valued uniformly continuous function on $(X^{\mathbb{N}}, \mathbf{u})$.

Theorem 1.11 is a part of the following one in which we summarize also some known results from [23, 37].

Theorem 6.5. *Let X be a compact connected metrizable Abelian group. Then*

- (i) ([23]) $\mathfrak{F}_0(X)$ is a connected Polish Abelian group.
- (ii) $\mathfrak{F}_0(X)$ is monothetic.
- (iii) $\mathfrak{F}_0(X)$ is a non-Schwartz group.
- (iv) ([23, 37]) $\mathfrak{F}_0(X)$ has countable dual. More precisely, $\widehat{\mathfrak{F}_0(X)} = \widehat{X}^{(\mathbb{N})}$.
- (v) ([37]) $\mathfrak{F}_0(X)$ is reflexive.
- (vi) $\mathfrak{F}_0(X)$ does not respect any property from \mathfrak{P} . In particular, $\mathfrak{F}_0(X)$ does not have the Schur property.
- (vii) $\mathfrak{F}_0(X)$ has the property UB.
- (viii) $c_0(\mathfrak{F}_0(X))$ is \mathfrak{g} -closed in $\mathfrak{F}_0(X)^{\mathbb{N}}$.

Proof. (i). The group $\mathfrak{F}_0(X)$ is connected and Polish by Fact 1.6. (ii) follows from Proposition 6.3. (iii). The group $\mathfrak{F}_0(X)$ is not a Schwartz group by Theorem 1.10. (vi) follows from Theorem 1.10. (vii) follows from Proposition 6.4.

(viii). To prove that $c_0(\mathfrak{F}_0(X))$ is \mathfrak{g} -closed in $\mathfrak{F}_0(X)^{\mathbb{N}}$ it is enough to show that for every $(\mathbf{y}_n)_{n \in \mathbb{N}} \in \mathfrak{F}_0(X)^{\mathbb{N}} \setminus c_0(\mathfrak{F}_0(X))$ there exists a sequence $\{\chi_l\}_{l \in \mathbb{N}} \in \widehat{\mathfrak{F}_0(X)}^{(\mathbb{N})}$ such that

- (1) $(\chi_l, \mathbf{x}_n) \rightarrow 1$ at $l \rightarrow \infty$, $\forall (\mathbf{x}_n) \in c_0(\mathfrak{F}_0(X))$, and
- (2) $(\chi_l, \mathbf{y}_n) \not\rightarrow 1$ at $l \rightarrow \infty$.

Let ρ be an invariant metric for X and $\mathbf{y}_n = (y_k^n)_{k \in \mathbb{N}} \in c_0(X)$. Since $\mathbf{y}_n \not\rightarrow 0$ in $\mathfrak{F}_0(X)$ we can find an increasing sequence $\{n_l\}_{l \in \mathbb{N}}$ such that

$$d(0, \mathbf{y}_{n_l}) = \sup_{k \in \mathbb{N}} \rho(0, y_k^{n_l}) > \delta > 0, \quad \forall l \in \mathbb{N}.$$

So, for every $l \in \mathbb{N}$ we can choose $k_l \in \mathbb{N}$ such that

$$(6.9) \quad \rho(0, y_{k_l}^{n_l}) > \delta.$$

Since X is compact and metrizable, without loss of generality we can assume that $y_{k_l}^{n_l}$ converges to an element $y \in X$. It follows from (6.9) that $\rho(0, y) \geq \delta > 0$. In particular, $y \neq 0$. Choose $g \in \widehat{X}$ such that

$$(6.10) \quad (g, y) \neq 1.$$

Set

$$\chi_l = \nu_{n_l}(\mathbf{g}_l) \in \widehat{\mathfrak{F}_0(X)}^{(\mathbb{N})}, \quad \text{where } \mathbf{g}_l = \nu_{k_l}(g) \in \widehat{\mathfrak{F}_0(X)}.$$

Let us check (1) and (2).

Fix $(\mathbf{x}_n) \in c_0(\mathfrak{F}_0(X))$, where $\mathbf{x}_n = (x_k^n)_{k \in \mathbb{N}} \in c_0(X)$. This means that

$$(6.11) \quad d(0, \mathbf{x}_n) = \sup_{k \in \mathbb{N}} \rho(0, x_k^n) \rightarrow 0 \text{ at } n \rightarrow \infty.$$

It follows from (6.11) that $x_{k_l}^{n_l} \rightarrow 0$ in X . Hence

$$(\chi_l, \mathbf{x}_n) = (\mathbf{g}_l, \mathbf{x}_{n_l}) = (g, x_{k_l}^{n_l}) \rightarrow 1, \text{ at } l \rightarrow \infty.$$

This proves (1). The inequality (6.10) yields

$$(\chi_l, (\mathbf{y}_n)) = (\mathbf{g}_l, \mathbf{y}_{n_l}) = \left(g, y_{k_l}^{n_l}\right) \rightarrow (g, y) \neq 1, \text{ at } l \rightarrow \infty,$$

that proves (2). Thus $c_0(\mathfrak{F}_0(X))$ is \mathfrak{g} -closed in $\mathfrak{F}_0(X)^{\mathbb{N}}$. \square

7. THE GLICKSBERG AND SCHUR PROPERTIES AND k - AND s -GROUPS

In this section we show that the Glicksberg and Schur properties can be naturally defined by two natural functors in \mathbf{TG} . For this we consider two important classes of topological groups introduced by Noble in [53, 54], namely, k - and s -groups.

(1) *The Glicksberg property and k -groups.* For every $(X, \tau) \in \mathbf{TG}$ denote by $k(\tau)$ the finest group topology for X coinciding on compact sets with τ . In particular, τ and $k(\tau)$ have the same family of compact subsets. Clearly, $\tau \leq k(\tau)$. If $\tau = k(\tau)$, the group (X, τ) is called a k -group [54]. The group $(X, k(\tau))$ is called the k -modification of X . The assignment $\mathbf{k}(X, \tau) := (X, k(\tau))$ is a coreflector from \mathbf{TG} to the full subcategory \mathbf{K} of all k -groups. The class \mathbf{K} contains all topological groups whose underlying space is a k -space. In particular, the class \mathbf{LC} of all locally compact groups is contained in \mathbf{K} . Since every metrizable group is a k -space we have $\mathbf{LC} \subsetneq \mathbf{K}$. The family of all Abelian k -groups we denote by \mathbf{KA} .

Denote by \mathbf{PCom} the class of all precompact groups, and by \mathfrak{RC} the class of all MAP groups which respect compactness. The item (1) in the next proposition shows that respect compactness is naturally defined by the functors \mathbf{k} and \mathfrak{B} . Note also that item (5) generalizes [67, Theorem 1.2] (see also Remark in [9]).

Proposition 7.1. *Let X and Y be MAP topological groups.*

- (1) $X \in \mathfrak{RC}$ if and only if $(\mathbf{k} \circ \mathfrak{B})(X) = \mathbf{k}(X)$.
- (2) $X \in \mathbf{K} \cap \mathfrak{RC}$ if and only if $(\mathbf{k} \circ \mathfrak{B})(X) = X$.
- (3) $\mathbf{PCom} \subsetneq \mathfrak{RC}$ and $\mathbf{LCA} \subsetneq \mathbf{KA} \cap \mathfrak{RC}$.
- (4) $\mathbf{K} \cap \mathfrak{RC} \subsetneq \mathbf{K}$ and $\mathbf{K} \cap \mathfrak{RC} \subsetneq \mathfrak{RC}$.
- (5) Let $X \in \mathbf{K}$ and $Y \in \mathfrak{RC}$ and let $\phi : X \rightarrow Y$ be a homomorphism. If $\phi^+ : X^+ \rightarrow Y^+, \phi^+(x) := \phi(x)$, is continuous, then ϕ is continuous.

Proof. (1). If $X \in \mathfrak{RC}$, then $(\mathbf{k} \circ \mathfrak{B})(X) = \mathbf{k}(X)$ by the definition of respect compactness and the definition of $\mathbf{k}(X)$.

Conversely, let $(\mathbf{k} \circ \mathfrak{B})(X) = \mathbf{k}(X)$ and K is compact in $\mathfrak{B}(X)$. Then K is compact in $(\mathbf{k} \circ \mathfrak{B})(X)$ by the definition of k -modification. So K is compact in $\mathbf{k}(X)$. Hence, by the definition of k -modification, K is compact in X . Thus $X \in \mathfrak{RC}$.

(2). Let $X \in \mathbf{K} \cap \mathfrak{RC}$. By item (1) and the definition of k -groups, we obtain $(\mathbf{k} \circ \mathfrak{B})(X) = \mathbf{k}(X) = X$.

Conversely, let $(\mathbf{k} \circ \mathfrak{B})(X) = X$. Since $\mathbf{k} \circ \mathbf{k} = \mathbf{k}$, the equalities

$$\mathbf{k}(X) = \mathbf{k} \circ (\mathbf{k} \circ \mathfrak{B})(X) = (\mathbf{k} \circ \mathfrak{B})(X) = X$$

and item (1) imply that X is a k -group and $X \in \mathfrak{RC}$.

(3). Since $\mathfrak{B}(K) = K$ for each precompact group K , the first inclusion follows. The second one holds by the Glicksberg theorem. To prove that these inclusions are strict take an arbitrary compact totally disconnected metrizable group X . Then $\mathfrak{F}_0(X)$ is metrizable, and hence it is a k -group. Now Theorem 1.10 and Fact 1.6(iii) imply that $\mathfrak{F}_0(X)$ respects compactness and it is not locally precompact. Thus the inclusions are strict.

(4). Being metrizable the group $\mathfrak{F}_0(\mathbb{T})$ belongs to \mathbf{KA} . However, $\mathfrak{F}_0(\mathbb{T})$ does not respect compactness by Theorem 1.10. Thus $\mathbf{K} \cap \mathfrak{RC} \neq \mathbf{K}$.

To prove that the second inclusion is strict it is enough to find a precompact Abelian group X which is not a k -group. Take an arbitrary non-measurable subgroup H of the circle \mathbb{T} and set $X := (\mathbb{Z}, T_H)$. Then the precompact group X does not contain non-trivial convergent sequences (see [15]). Since X is countable, we obtain that X also has no infinite compact subsets by [31, 3.1.21]. This immediately implies that the

k -modification $\mathbf{k}(X)$ of X is discrete. Hence $\mathbf{k}(X) = \mathbb{Z}_d$ is a discrete LCA group. So $\mathbf{k}(X) \neq X$ and X is not a k -group. Thus the second inclusion is strict.

(5). Let $id_X : X \rightarrow X^+$ and $id_Y : Y \rightarrow Y^+$ be the identity continuous maps. Fix arbitrarily a compact subset K in X . Then $K^+ := \phi^+(id_X(K))$ is compact in Y^+ . As $Y \in \mathfrak{RC}$, K^+ is compact in Y . So $id_Y|_{K^+}$ is a homeomorphism. Hence $\phi|_K = (id_Y|_{K^+})^{-1} \circ \phi^+ \circ (id_X|_K)$ is continuous. So ϕ is k -continuous. As X is a k -group, ϕ is continuous (see [54]). \square

Remark 7.2. In [19], the authors show that the answer to both questions posed in [19, 1.2] (see also [67]) is “no”. Let us show that the group X in the proof of item (4) of Proposition 7.1 also answers negatively to those questions. We use the notation from [19, 1.2]. Set $G = \mathbb{Z}$ and $\mathcal{U} = T_H$. Since G is countable, every locally compact group topology \mathcal{T} on G must be discrete. So $\mathcal{T}^+ = T_{\mathbb{T}}$. Further, as it was noticed in item (4), a subset A of G is \mathcal{T} -compact if and only if A is \mathcal{U} -compact (if and only if A is finite). However, since $H \neq \mathbb{T}$, we obtain $\mathcal{U} \neq \mathcal{T}^+$ by Fact 2.10.

We do not know an answer to the next questions:

Problem 7.3. Let $X \in \mathbf{KA}$ (or just a k -space). Is $\mathfrak{F}_0(X)$ a k -group?

Problem 7.4. Let X be a LCA group. Is $\mathfrak{F}_0(X)$ a k -space?

(II) *The Schur property and s -groups.* Similar to k -groups we define s -groups (we follow [34]). Let (X, τ) be a (Hausdorff) topological group and let S be the set of all sequences in (X, τ) converging to the unit. Then there exists the finest Hausdorff group topology τ_S on the underlying group X in which all sequences of S converge to the unit. If $\tau = \tau_S$, the group X is called an s -group. The assignment $\mathbf{s}(X, \tau) := (X, \tau_S)$ is a coreflector from \mathbf{TG} to the full subcategory \mathbf{S} of all s -groups. The class \mathbf{S} contains all sequential groups [34, 1.14]. Every s -group is also a k -group [35]. Note that X and $\mathbf{s}(X)$ have the same set of convergent sequences [34, 4.2]. The family of all Abelian s -groups we denote by \mathbf{SA} .

Denote by \mathfrak{RG} the class of all MAP groups which have the Schur property. In analogy to Proposition 7.1 we obtain:

Proposition 7.5. Let X and Y be MAP topological groups.

- (1) $X \in \mathfrak{RG}$ if and only if $(\mathbf{s} \circ \mathfrak{B})(X) = \mathbf{s}(X)$.
- (2) $X \in \mathbf{S} \cap \mathfrak{RG}$ if and only if $(\mathbf{s} \circ \mathfrak{B})(X) = X$.
- (3) $\mathbf{S} \cap \mathfrak{RG} \subsetneq \mathbf{S}$ and $\mathbf{S} \cap \mathfrak{RG} \subsetneq \mathfrak{RG}$.
- (4) Let $X \in \mathbf{S}$ and $Y \in \mathfrak{RG}$ and let $\phi : X \rightarrow Y$ be a homomorphism. If $\phi^+ : X^+ \rightarrow Y^+, \phi^+(x) := \phi(x)$, is continuous, then ϕ is continuous.

Proof. (1). If $X \in \mathfrak{RG}$, then $(\mathbf{s} \circ \mathfrak{B})(X) = \mathbf{s}(X)$ by the definition of respect sequentiality and the definition of $\mathbf{s}(X)$.

Conversely, let $(\mathbf{s} \circ \mathfrak{B})(X) = \mathbf{s}(X)$ and $x_n \rightarrow e$ in X^+ . Then $x_n \rightarrow e$ in $(\mathbf{s} \circ \mathfrak{B})(X)$ by [34, Lemma 4.2], and hence $x_n \rightarrow e$ in $\mathbf{s}(X)$. So $x_n \rightarrow e$ in X by [34, Lemma 4.2]. Thus $X \in \mathfrak{RG}$.

(2). Let $X \in \mathbf{S} \cap \mathfrak{RG}$. By item (1) and the definition of s -groups, we obtain $(\mathbf{s} \circ \mathfrak{B})(X) = \mathbf{s}(X) = X$.

Conversely, let $(\mathbf{s} \circ \mathfrak{B})(X) = X$. Since $\mathbf{s} \circ \mathbf{s} = \mathbf{s}$, the equalities

$$\mathbf{s}(X) = \mathbf{s} \circ (\mathbf{s} \circ \mathfrak{B})(X) = (\mathbf{s} \circ \mathfrak{B})(X) = X$$

and item (1) imply that X is an s -group and $X \in \mathfrak{RG}$.

(3). Being metrizable the group $\mathfrak{F}_0(\mathbb{T})$ belongs to \mathbf{SA} . However, $\mathfrak{F}_0(\mathbb{T})$ does not respect sequentiality by Theorem 1.10. Thus $\mathbf{S} \cap \mathfrak{RG} \neq \mathbf{S}$. The example in the proof of Proposition 7.1(4) shows also that the second inclusion is strict.

(4). Let $id_X : X \rightarrow X^+$ and $id_Y : Y \rightarrow Y^+$ be the identity continuous maps. Fix arbitrarily a convergent sequence \mathbf{u} with the limit point in X (so \mathbf{u} is a compact subset of X). Then $\mathbf{u}^+ := \phi^+(id_X(\mathbf{u}))$ is a convergent sequence with the limit point in Y^+ . As $Y \in \mathfrak{RG}$, \mathbf{u}^+ is a convergent sequence in Y . So $id_Y|_{\mathbf{u}^+}$ is a homeomorphism. Hence $\phi|_{\mathbf{u}} = (id_Y|_{\mathbf{u}^+})^{-1} \circ \phi^+ \circ (id_X|_{\mathbf{u}})$ is continuous. So ϕ is a sequentially continuous homomorphism. As X is an s -group, ϕ is continuous [34]. \square

The next question is open:

Problem 7.6. *Let $X \in \mathbf{SA}$ (in particular, the group X is sequential). Is $\mathfrak{F}_0(X)$ an s -group?*

8. OPEN QUESTIONS

In this section we define another functors naturally coming from Functional Analysis and pose some open questions.

(I) *Functors \mathfrak{F}_0^I , \mathfrak{F}_∞^I and \mathfrak{F}^I on \mathbf{TAG} .* Let I be an arbitrary infinite set of indices and X be an Abelian topological group. The collection $\{U^I : U \in \mathcal{N}(X)\}$ forms a base at 0 for a group topology \mathfrak{u} in X^I . We call \mathfrak{u} the *uniform topology*.

We denote by $c_0^I(X)$ the set of all $(x_i)_{i \in I} \in X^I$ such that, for every $U \in \mathcal{N}(X)$, $x_i \in U$ for all but finitely many indices. The set of all $(x_i)_{i \in I} \in X^I$ such that the set $\{x_i\}_{i \in I}$ is precompact in X , we denote by $\ell_\infty^I(X)$.

The uniform group topology on $c_0^I(X)$ and $\ell_\infty^I(X)$ induced from (X^I, \mathfrak{u}) we denote by \mathfrak{u}_0 and \mathfrak{u}_∞ respectively. In analogy to \mathfrak{F}_0 we define the functors \mathfrak{F}_0^I , \mathfrak{F}_∞^I and \mathfrak{F}^I on the category \mathbf{TAG} by the assignment

$$\begin{aligned} X &\rightarrow \mathfrak{F}^I(X) := (X^I, \mathfrak{u}), \\ X &\rightarrow \mathfrak{F}_0^I(X) := (c_0^I(X), \mathfrak{u}_0), \\ X &\rightarrow \mathfrak{F}_\infty^I(X) := (\ell_\infty^I(X), \mathfrak{u}_\infty). \end{aligned}$$

In the case $I = \mathbb{N}$ we shall omit the subscript I . If $X = \mathbb{R}$, then $\mathfrak{F}_\infty(\mathbb{R})$ coincides with the classical Banach space ℓ^∞ . On the other hand, the group $\mathfrak{F}(\mathbb{R})$ is not a TVS. Clearly, \mathfrak{F}^I and \mathfrak{F}_∞^I coincide on precompact groups.

It would be interesting to consider Problems 1.1-1.5 for these functors. Let us note that it is not clear even whether $\mathfrak{F}_\infty^I(X)$ is connected for a compact connected group X , although it is known that $\mathfrak{F}_\infty(\mathbb{T})$ is even arc-connected. Clearly, to obtain an analogue to Fact 1.8 and Theorem 1.10 we should describe the dual group of $\mathfrak{F}_\infty^I(X)$.

Problem 8.1. *Let X be a (compact) LCA group. Describe $\widehat{\mathfrak{F}_\infty^I(X)}$ or $\widehat{\mathfrak{F}_\infty(X)}$.*

Note that these groups are very “big”. For example, it is known that $\text{Card}(\widehat{\mathfrak{F}_\infty(\mathbb{T})}) = 2^c$ (see [24]).

However, even for the simplest case $X = \mathbb{T}$ we know only the cardinality of the dual group $\widehat{\mathfrak{F}_\infty(\mathbb{T})}$, but not its appropriate description (as, for example, in Fact 2.22).

The following question is of interest:

Problem 8.2. *Let X be a (compact) LCA group and I be an infinite set. Are the groups $\mathfrak{F}_0^I(X)$, $\mathfrak{F}_\infty^I(X)$ and $\mathfrak{F}^I(X)$ reflexive?*

By Fact 1.8 we know only that $\mathfrak{F}_0(X)$ is reflexive.

(II) Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{P}$ be distinct properties and let Γ be an arbitrary family of Abelian topological groups. We say that $\mathcal{P}_1 \leq_R \mathcal{P}_2$ on Γ if every $X \in \Gamma$ which respects \mathcal{P}_1 respects also \mathcal{P}_2 . If $\mathcal{P}_1 \leq_R \mathcal{P}_2$ and $\mathcal{P}_2 \leq_R \mathcal{P}_1$ on Γ we say that \mathcal{P}_1 and \mathcal{P}_2 are *equivalent* on Γ . So, all properties from \mathfrak{P} are equivalent on \mathbf{LCA} by [67], and all properties from \mathfrak{P}_0 are equivalent on the class of all nuclear groups by [9]. Further, the properties $\mathcal{C}, \mathcal{CC}, \mathcal{PC}$ and \mathcal{FB} are equivalent on the class of all complete Abelian g -groups by [47, Theorem 3.3]. These results justify the next question (cf. [2, Question 4.6]):

Problem 8.3. *Are the properties from \mathfrak{P} or \mathfrak{P}_0 equivalent on the class of all (complete) MAP locally quasi-convex Schwartz groups?*

Theorem 4.6 says that the answer to this question is “yes” if the next question has the positive answer:

Problem 8.4. *Let X be a (complete) MAP locally quasi-convex Schwartz groups. Does X have the SB-property?*

As it was mentioned, there are Schur groups without the Glicksberg property [70] (see a complete proof in [29, Example 19.19]). Taking into account also Diagram 4.1 it seems to be interesting the next question.

Problem 8.5. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{P}$ be distinct properties and $\mathcal{P}_1 \leq_R \mathcal{P}_2$ on MAPA. Find a MAP Abelian group X which respects \mathcal{P}_2 but which does not respect \mathcal{P}_1 .*

It would be interesting also to consider Problems 1.5, 7.3 and 7.6 for other classes of MAP Abelian groups (as, for instance, Polish, nuclear or Schwartz MAP groups).

In connection with Fact 1.7 and Theorem 1.11(iii) the next problem is of interest.

Problem 8.6. *Let X be an Abelian topological group such that $c_0(X)$ is \mathfrak{g} -closed in $X^{\mathbb{N}}$. Under which conditions the group X has the Schur property?*

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