$(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ **IS A GELFAND PAIR**

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ABSTRACT. We prove that $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ is a Gelfand pair. More precisely, we show that for an irreducible smooth admissible Frechet representation (π, E) of $GL_{2n}(\mathbb{C})$ the space of continuous functionals $Hom_{Sp_{2n}(\mathbb{C})}(E, \mathbb{C})$ is at most one dimensional. For this we show that any distribution on $GL_{2n}(\mathbb{C})$ invariant with respect to the double action $Sp_{2n}(\mathbb{C}) \times Sp_{2n}(\mathbb{C})$ is transposition invariant. Such a result was previously proven for p-adic fields by M. Heumos and S. Rallis.

1. INTRODUCTION

Let F be a p-adic field (i.e. a local field of characteristic zero). It was proven in [HR] that $(GL_{2n}(F), Sp_{2n}(F))$ is a Gelfand pair. A simple proof, for the case where F is a finite field, appeared recently in [GG]. In [OS3] we used Proposition 3.1 of [GG] to simplify the proof of the p-adic case (the idea of proof, namely the usage of the Gelfand Kazhdan method remains the same).

Our goal in this note is to transfer the result to the archimedean case. This is the first step in extending the joint work of Omer Offen and the author (see [OS1], [OS2], [OS3]) to the archimedean case.

More precisely the main result of the present note is the following:

Theorem (A). *Let (π, E) be a irreducible admissible smooth Frechet representation (π, E) of $GL_{2n}(\mathbb{C})$. Then the space of functionals $Hom_{Sp_{2n}(\mathbb{C})}(E, \mathbb{C})$ is at most one dimensional.*

Following the standard technique of Gelfand and Kazhdan we first show

Theorem (B). *Let T be a distribution on $GL_{2n}(\mathbb{C})$ invariant with respect to the double action $Sp_{2n}(\mathbb{C}) \times Sp_{2n}(\mathbb{C})$. Then T is transposition invariant.*

For the proof we will use an analogue of Gelfand-Kazhdan criterion as proved in [AGS] as well as the strategy of [AG2]-[AG3]. Namely, we verify that the symmetric pair $GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ is **good** and that all its **descendants** are **regular** (See section 2 for the exact definitions of these terms). We hope to cover the case $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ in the near future.

We have made an attempt to make this note as self contained as possible. Since our proof relies heavily on the notions and ideas of [AGS] and [AG2] we summarized those in section 2. We introduce the notion of Gelfand pair and review the Gelfand-Kazhdan distributional technique.

We then review some notions and results of [AG2], especially the notions of symmetric pair, descendants of a symmetric pair, good symmetric pair and regular symmetric pair.

The formal argument is given in the section 3 while the key computation of descendants of our symmetric pair is done in section 4.

2. Notations and Preliminaries on Gelfand Pairs

In this section we review some notions and result concerning Gelfand pairs. We follow the notions introduced in [AGS] and [AG2].

2.1. **Notation.** We let F be a p-adic field, i.e. a local field of characteristic zero. We denote algebraic F-varieties by bold face letters. Thus $\mathbf{X}$ is an F-variety and $X = \mathbf{X}(F)$ is the set of its F-points. Note that when F is non-archimedean X is an ℓ-space (as in [BZ]) while for archimedean F the set X has a structure of smooth manifold. We consider the space $S^*(X)$ of Schwartz distributions on X. For F non-archimedean these are just linear functionals on the space $S(X)$ of locally constant functions with compact support on the ℓ-space X. When F is archimedean we consider X as a Nash manifold and use the Schwartz space of [AG1]. We note that the main result of the paper, Theorem B, holds true when one replaces $S^*(X)$ with $\mathcal{D}(X)$ the space of all distributions on X (see Theorem 4.0.8 of [AG2]).

The letters $\mathbf{G}, \mathbf{H}$ denotes reductive algebraic groups defined over F. The groups of points of which will be denoted by G, H respectively.

When we consider symmetric pairs we refer to triples $\mathbf{X} = (\mathbf{G}, \mathbf{H}, \theta)$ of reductive F-groups where $\theta : \mathbf{G} \to \mathbf{G}$ is an involution and $\mathbf{H} = \mathbf{G}^\theta$ is the fixed point of θ. The anti-involution $\sigma(g) = \theta(g^{-1})$ and the symmetrization map $s(g) = \sigma(g)g \in G^\sigma$ will play a key role.

For a symmetric pair $\mathbf{X}$ we will denote by $X(F)$ the pair of groups (G, H) of corresponding F-points of the groups $\mathbf{G}, \mathbf{H}$.

If A, B are matrices we will use the notation $diag(A, B)$ for the block matrix whose blocks are A and B.

For $n \geq 1$ we let $G_n = GL(n)$ and let J_{2n} be the non degenerate skew symmetric matrix of size $2n$ given by

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

Clearly,

$$\theta_{2n}(g) = J_{2n}^{-1t}g^{-1}J_{2n}$$

is an involution on G_{2n} and we have

$$H_{2n} = G_{2n}^{\theta_{2n}} = \{g \in G_{2n} :^t gJ_{2n}g = J_{2n}\} = Sp(2n)$$

We record the anti-involution

$$\sigma_{2n}(g) = \theta_n(g^{-1}) = J_{2n}^{-1t}gJ_{2n}$$

and the symmetrization map

$$s(g) = gJ_{2n}^{-1t}gJ_{2n}$$

2.2. Distributional Criterion for Gelfand pairs.

Definition 2.2.1. *Let* **G** *be a reductive group. By an **admissible representation of** G we mean an admissible representation of $G(F)$ if F is non-archimedean (see* [BZ]*) and admissible smooth Fréchet representation of $G(F)$ if F is archimedean.*

We recall the following three notions of Gelfand pair.

Definition 2.2.2. *Let* $H \subset G$ *be a pair of reductive groups.*
- *We say that* (G, H) *satisfy* **GP1** *if for any irreducible admissible representation* (π, E) *of* G *we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \leq 1$$
- *We say that* (G, H) *satisfy* **GP2** *if for any irreducible admissible representation* (π, E) *of* G *we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \cdot \dim Hom_H(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that* (G, H) *satisfy* **GP3** *if for any irreducible* **unitary** *representation* $(\pi, \mathcal{H})$ *of* $G(F)$ *on a Hilbert space* $\mathcal{H}$ *we have*
$$\dim Hom_{H(F)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vD, BvD]).

Clearly,
$$GP1 \Longrightarrow GP2 \Longrightarrow GP3.$$

In [AGS] we obtained the following version of a classical theorem of Gelfand and Kazhdan.

Theorem 2.2.3. *Let* $H \subset G$ *be reductive groups and let* τ *be an involutive anti-automorphism of* G *and assume that* $\tau(H) = H$. *Suppose* $\tau(\xi) = \xi$ *for all bi* $H(F)$-*invariant Schwartz distributions* ξ *on* $G(F)$. *Then* (G, H) *satisfies GP2.*

In the case (G_{2n}, H_{2n}) at hand GP2 is equivalent to GP1 by the following simple proposition (see [AGS]).

Proposition 2.2.4. *Let* $H \subset GL(n)$ *be any transpose invariant subgroup. Then GP1 is equivalent to GP2 for the pair* $(GL(n), H)$.

2.3. Symmetric Pairs, GK pairs and Gelfand Pairs.
We now restrict to the case where $H \subset G$ is a symmetric subgroup and more precisely (G, H, θ) is a symmetric pair.

Definition 2.3.1. *A **symmetric pair** is a triple* (G, H, θ) *where* **H** $\subset$ **G** *are reductive groups, and* $\theta : \mathbf{G} \to \mathbf{G}$ *is an involution of* **G** *such that* **H** $=$ **G**$^\theta$.

In [AG2] the notion of **GK** symmetric pair is introduced. It means that the condition of the above criterion (Theorem 2.2.3) are met with τ being σ.

Definition 2.3.2. (G, H, θ) *is called* **GK** *pair if any* $H(F) \times H(F)$ *invariant distribution is* σ *invariant.*

In particular it follows from Theorem 2.2.3 that

$$GK \Longrightarrow GP2.$$

We now wish to describe the main tool to show that a given symmetric pair is GK pair. In principle there is one obvious obstruction for a pair to be GK pair. If there exist any closed orbit $\Delta = HgH$ which is NOT σ invariant then we can find a distribution supported on that orbit which is not σ invariant and this contradicts the condition. Thus an obvious necessary condition is the following:

(*) for each closed orbit HgH we have $\sigma(g) \in HgH$

Following [AG2], we call symmetric pairs satisfying condition (*) **good** symmetric pairs.

Definition 2.3.3. *The symmetric pair (G, H, θ) is called good if for all closed orbits $\Delta = HgH$ we have $\sigma(\Delta) = \Delta$.*

Thus,

$$GK \Longrightarrow good$$

In [AG2] it is conjectured that the converse is also true and a partial result in that direction is proved.

To formulate the theorem we need two notions, that of **descendant** symmetric pair and that of **regular** symmetric pair.

The exact technical notion of **regular** symmetric pairs will be explained later and for the moment we note that the expectation is that any symmetric pair is regular. Later we will discus this notion and explain how we will verify that certain symmetric pairs are regular.

The notion of **descendant** pairs is central for the present work. Given a symmetric pair $(\mathbf{G}, \mathbf{H}, \theta)$ we consider the centralizers in $\mathbf{G}$ and in $\mathbf{H}$ of certain semi simple elements $x \in G(F)$. The elements x that are considered are in the image of the symmetrization map and in particular satisfy $\sigma(x) = x$.

Being the centralizers of semi-simple elements these groups $\mathbf{H_x} \subset \mathbf{G_x}$ are reductive, they are defined over F and H_x is a symmetric subgroup of G_x, namely the fixed point of $\theta_x = \theta|_{G_x}$.

Let us now describe the elements $x \in G(F)$ that we consider. Given $g \in G$ such that $\Delta(g) = HgH$ is closed it is known that $x = s(g)$ is semi-simple and the stabilizer of x in H is exactly the stabilizer of g in $H \times H$ (for a proof see Proposition 7.2.1 of [AG2]). One can define a baby symmetric pair $(G_x, H_x, \theta|_{G_x})$ by taking the centralizers of x in G and H.

This is called a **descendant** of (G, H, θ). Notice that the set of descendants of a given symmetric pair depends on the field F over which the groups are considered.

Remark 2.3.4. *It follows from standard finiteness results that the number of non-conjugate descendants pairs of a given symmetric pair is always finite (see corollary 1, p.107 of [St]).*

We can now state one of the main results of [AG2]:

Theorem 2.3.5. *Suppose that the pair (G, H, θ) is **good** and that all its **descendants** are **regular**. Then (G, H, θ) is a **GK** pair.*

2.4. **The notion of regularity.** To define the notion of regular symmetric pair we need several more notations. Given an algebraic representation V of G we say that $v \in V(F)$ is nilpotent if zero is in the closure of its orbit $G(F)v$. We denote by $\Gamma(V)$ the set of nilpotent elements.

Consider the $Q(V) = (V/V^G)(F)$ and the subspace $R(V) \subset Q(V)$ of regular element (i.e. those that are not nilpotent). Notice that since G is reductive we can realize $Q(V)$ as a subspace of $V(F)$. Thus,

$$Q(V) = V/V^G$$

and

$$R(V) = Q(V) - \Gamma(V)$$

where $\Gamma(V)$ is the set of nilpotent elements for the action of G on V.

Let (G, H, θ) be a symmetric space.

We now formulate two properties $Dist_{gen}(g)$ and $Dist_{all}(g)$.

$$Dist_{gen}(g, V) \Longleftrightarrow S^*(R(V))^{H(F)} \subset S^*(R(V))^{Ad(g)}$$

and

$$Dist_{all}(g, V) \Longleftrightarrow S^*(Q(V))^{H(F)} \subset S^*(Q(V))^{Ad(g)}$$

We remind that $S^*(Q(V))$ is the usual space of Schwartz distributions on a vector space, while for the definition of $S^*(R(V))$ we use the theory developed in [AG1].

Now consider $\mathfrak{g} := LieG$ and the adjoint action of $H(F)$ on the vector space $\mathfrak{g}^\sigma := \{a \in \mathfrak{g} | \theta(a) = -a\}$.

We expect that for some elements $g \in G(F)$ the fact that each $H(F)$-invariant distribution on the regular part $R(V) \subset Q(V)$ is $Ad(g)$ invariant will imply that the same is true for distributions on the entire vector space $Q(V)$.

We now specify the elements $g \in G(F)$ that will play a role in the definition of regularity. We deviate here slightly from [AG2]. Let $Z = Z(G)$ be the center of G.

An element $g \in G$ is called **relevant** if $s(g) := g\sigma(g) \in Z(G)$ and H has index at most 2 in the group $H < g >$ generated in $Aut(G)$ by $Ad(H)$ and $Ad(g)$.

We denote by

$$rel(G, H, \theta) = \{g \in G : s(g) \in Z(G), [H < g >: Ad(H)] \le 2\}$$

the set of relevant elements.

We can now give the definition of regularity.

Definition 2.4.1. *The pair (G, H) is called* **regular** *if for the adjoint action of $H(F)$ on the vector space $\mathfrak{g}^\sigma$*

$$Dist_{gen}(g, \mathfrak{g}^\sigma) \Longrightarrow Dist_{all}(g, \mathfrak{g}^\sigma)$$

for all relevant elements $g \in rel(G, H, \theta)$.

Remark 2.4.2. *In* [AG2] *the authors use the notion of admissible elements. These satisfy a further condition which is implied by $Dist_{gen}(g, \mathfrak{g}^\sigma)$. Namely, if $Dist_{gen}(g, \mathfrak{g}^\sigma)$ holds then each closed $H(F)$ orbit on $\mathfrak{g}^\sigma$ is preserved by $Ad(g)$. This is exactly the condition $Ad(g)|_{\mathfrak{g}^\sigma}$ is* H-**admissible**. *Thus the definition given here for regularity is identical to that of* [AG2].

We have the following trivial consequence of the definitions.

Proposition 2.4.3. *We have the following:*
 (i) the product of regular pairs is a regular pair.
 (ii) Let (G, H, θ) be a symmetric pair. $Z = Z(G)$ the center of G. Suppose that

$$\{g : \sigma(g)g \in Z\} \subset ZH.$$

Then (G, H, θ) is regular.

Proof. The first point is proposition 7.4.4 of [AG2] (We emphasize that this is a direct consequence of the definitions and the Proposition 3.1.5 of [AGS]).

For the second point we note that $rel(G, H, \theta)$ is by assumptions a subset of ZH and hence the condition $Dist_{all}(g)$ is obviously satisfied. $\square$

3. $\mathbf{X_n}(F) := (G_{2n}, H_{2n}, \theta)$ SATISFIES GK WHEN $F \neq \mathbb{R}$

In this section we prove Theorem A and Theorem B.

For the proof of Theorem A it is enough to show that the pair (G_{2n}, H_{2n}, θ) is GK.

Indeed, by Theorem 2.2.3 it satisfies GP2 and hence, by Proposition 2.2.4 it satisfies GP1. This is the content of Theorem A.

Observe that Theorem B is equivalent to the assertion that the pair (G_{2n}, H_{2n}, θ) is GK because the anti-involution σ_{2n} is conjugate to transposition via the element J_{2n} of Sp_{2n}.

To show that X_n is GK we will verify that it is **good** and compute its descendants.

When $F = \mathbb{C}$ we show that all these descendants are regular. Thus by Theorem 2.3.5 the pair $X_n(\mathbb{C})$ is GK.

Remark 3.0.4. *It can be shown that $X_n(F)$ is regular also for F non-archimedean. We don't need this fact here since our proof that $X_n(F)$ is GK will be based on the original Gelfand-Kazhdan criterion that requires the verification that **all** the orbits $\Delta = HgH$ are σ-invariant. See below.*

3.1. **The pair $X_n(F)$ is good.** We first verify that all orbits are transpose invariant.

Proposition 3.1.1. *Let F be an arbitrary field. Let $g \in GL_{2n}(F)$. Then $g^t \in HgH$ where $H = Sp_{2n}(F)$.*

The proof depends on the following lemma (which is proposition 3.1 of [GG]):

Lemma 3.1.2. *Let F be an arbitrary field. For $x \in GL_n(F)$ define $d(x) = diag(x, I_n)$. The map $Ad(G_n)x \to H_{2n}d(x)H_{2n}$ is a bijection between the set of conjugacy classes in $GL_n(F)$ and the set of orbits of $Sp_{2n}(F) \times Sp_{2n}(F)$ in $GL_{2n}(F)$. In particular, given $g \in GL_{2n}(F)$ the double coset $H_{2n}gH_{2n}$ contains an element of the form $d(x) = diag(x, I_n)$ where $x \in GL_n(F)$.*

Proof of proposition 3.1.1
Since H is transpose invariant, the statement is equivalent to

$$HgH = (HgH)^t$$

Thus we can choose whatever representative we like in HgH. We take $g = diag(x, I_n)$ as in the lemma. But for some $k \in GL(n)$ we have $kxk^{-1} = x^t$ since in $GL(n)$ any matrix is

conjugate to its transposed. Taking $h = diag(k, {}^t k^{-1})$ which belongs to H_{2n} we have

$$h \bullet diag(x, I_n) \bullet h^{-1} = diag({}^t x, I_n)$$

as needed.

Corollary 3.1.3. *Let F be a p-adic field. Then the pair $(GL_{2n}(F), Sp_{2n}(F))$ is good.*

Indeed, by the above proposition and $J_{2n} \in H_{2n}$ we see that for each orbit $\Delta = HgH$ we have

$$\sigma(g) = J^t g J \in HgH$$

Remark 3.1.4. *For the case $F = \mathbb{C}$, it is shown in [AG2] (see corollary 7.1.7) that any connected symmetric pair (i.e. G/H is connected) is* **good**.

Remark 3.1.5. *The fact that the pair $X_n(F)$ is good for any local field F can be deduced from the* **cohomological criterion** *given in corollary 7.1.5 of [AG2]. Indeed, all descendants are products of symplectic groups (as shown in section 4 of this note) and these are known to be cohomologically trivial ([Sp]). In a sense the approach of [HR] is along these lines.*

Since all orbits of $H_{2n}(F) \times H_{2n}(F)$ on $G_{2n}(F)$ are σ-invariant one can use, in the case that F is a p-adic field the work of Gelfand-Kazhdan to deduce

Theorem 3.1.6. *Let F be a non-archimedean field. Then the pair (G_{2n}, H_{2n}) is a GK and hence a Gelfand pair.*

The complete argument can be extracted from page 6 of [OS3] but it is really an immediate consequence of the theory of Gelfand-Kazhdan as presented by in [BZ].

Remark 3.1.7. *Note that when F is non-archimedean the condition of regularity is not required in our case since* **all the orbits** *for the pair X_n are σ-invariant.*

3.2. **The descendants of the pair** $(GL_{2n}(F), Sp_{2n}(F))$. We have

Proposition 3.2.1. *All descendants of $X_n(F)$ are products of symmetric pairs of the form $X_m(E)$ for some $m \leq n$ and some E/F finite extension.*

For the proof see the fourth section where the proposition is formulated in slightly different form.

3.3. **Regularity of X_n and its descendants,** $F = \mathbb{C}$. We now restrict to the case $F = \mathbb{C}$.

Corollary 3.3.1. *Let $F = \mathbb{C}$. The symmetric pair $X_n(F)$ and all its descendants are regular.*

Proof. By proposition 3.2.1, each descendant is a product of $X_m(E)$ for some field extension E/F and $m < n$. Thus, by clause (i) of proposition 2.4.3, it is enough to prove that $X_m(E)$ is regular for any E/F and any m.

Let g be a relevant element. Then $g\sigma(g) \in Z(G_{2m})$ and since $F = \mathbb{C}$ we can find a scalar matrix $\alpha \in Z(G_{2m})$ and $g_0 \in H_{2m}$ such that $g = \alpha g_0$. Thus we search for $\alpha \in Z(G_{2m})$ such that $\alpha^{-1}g \in H$. We claim that any α solving the equation

(1) $$\alpha\sigma(\alpha) = z$$

Will work. Indeed,

$$\alpha^{-1}g\sigma(g)\sigma(\alpha)^{-1} = 1$$

(we use $z = g\sigma(g)$) and thus $s(\alpha^{-1}g) = 1$ showing that $\alpha^{-1}g \in H$. We thus need to solve the equation (1) or

$$\alpha J^{-1}\alpha J = z$$

This can be done by hands after writing $z = diag(t, ..., t)$ and taking $\alpha = diag(r, r, .., r)$ with $r = t^{1/2}$. By clause (ii) of proposition 2.4.3 we are done.   □

4. The descendants of the pair $(GL(V), Sp(V))$

Let F be arbitrary local field. Let (V, ω) be a symplectic space over F. Let $J : V \to V^*$ be the isomorphism induced by ω. J is given by $J(v)(u) = \omega(v, u)$. Clearly, $J^* = -J$ where J^* is the adjoint operator to J.

Consider the involution $\theta(g) = J^{-1}g^\tau J$ where g^τ is the inverse of the adjoint of $g : V \to V$. Note that the isomorphism $\tau : GL(V) \to GL(V^*)$ given by $\tau(g) = (g^*)^{-1}$ correspond in matrix notations to transpose inverse.

The fixed point group of θ is the symplectic group

$$Sp(V, J) = \{g \in GL(V) : Jg = g^\tau J\}$$

Clearly, this is the group of automorphisms of the symplectic space (V, ω). The symmetric pair $(GL(V), Sp(V, J), \theta)$ will be denoted $X_{V,J}$.

The aim of this section is to verify the following

Theorem 4.0.2. *Let F be arbitrary local field. Let $X_{V,J}$ be the symmetric pair as above. Then all descendants of $X_{V,J}$ are products of pairs of the form $X_{W,I}$ where (W, I) is a symplectic space over a finite extension E/F with $dim_E(W) \le dim_F(V)$.*

For the proof of the main result of the note we need the above only for the case $F = \mathbb{C}$. In that case it can be proven by brute force. One can treat arbitrary local field using the method of [SpSt] as we now show.

We recall the formula for the centralizer of a semi-simple transformation. Remind that for a semi-simple element $x \in GL(V)$ the minimal polynomial $P = min(x, F)$ is a product of distinct F-irreducible polynomials. Moreover, if $P = \prod_{i=1}^{n} P_i$ then

(2) $$V = \oplus V_i$$

with $V_i = Ker(P_i(x))$. Note that the adjoint operator x^t yields a decomposition $V^* = \oplus V_i^*$ where $V_i^* = Ker(P_i(x^*))$. Here $x^* : V^* \to V^*$ is defined by

$$x^*(\ell)(v) = \ell(xv).$$

We have $x^\tau = (x^*)^{-1}$.

Notice that each V_i is a vector space over the field $E_i = F[t]/(P_i(t))$. Also, an element that commutes with x must preserve the direct sum decomposition $V = \oplus V_i$ and clearly commutes with the action of each E_i on V_i. Thus,

$$GL(V)_x = \prod_{i=1}^{n} GL(V_i, E_i)$$

The content of proposition 2.8 (page E-89) in [SpSt] is an extension of this well known fact to describe the centralizers of elements of classical groups. We need a related result, namely a description of centralizers of elements that satisfy $x = \sigma(x)$ (as opposed to $x = \theta(x)$). Let us present the result in the case where V is endowed with a symplectic form ω or equivalently with $J : V \to V^*$ which is an isomorphism and anti-self dual. We remind that $\theta(x) = J^{-1}(x^*)^{-1}J$ and $\sigma(x) = J^{-1}x^*J$.

We will require two simple lemmas.

Lemma 4.0.3. *Suppose $x = \sigma(x)$. Then the decomposition (2) $V = \oplus V_i$ is orthogonal with respect to ω.*

Proof. First note that the condition $Jx = x^*J$ is equivalent to $\omega(xv, u) = \omega(v, xu)$. Let $v \in Ker(P(x))$ and $u \in ker(Q(x))$ with P, Q distinct factors and hence prime to each other. Find $a(x), b(x)$ with

$$v = a(x)P(x)v + b(x)Q(x)v.$$

We have

$$\omega(v, u) = \omega(b(x)Q(x)v, u) = \omega(b(x)v, Q(x)u) = 0$$

$\square$

Lemma 4.0.4. *Let $x \in GL(V)$ a semi-simple element with $x = \sigma(x)$. Let $P(x) = \prod_{i=1}^{n} P_i(x)$ be the decomposition of the minimal polynomial of x to irreducible factors. Then $J : V \to V^*$ maps each $V_i = ker(P_i(x))$ to $V_i^* = Ker(P_i(x^*))$. Moreover, J is E_i linear on each V_i and $J|_{V_i}$ defines a non-degenerate form on the space V_i.*

Proof. If $v \in V_i$ then $P_i(x)v = 0$. Now since $Jx = x^*J$ we get $P_i(x^*)Jv = JP_i(x)v = 0$ and hence $Jv \in V_i^*$. The E_i linearity is checked using the same identity. We thus see that $J = \oplus J_i$ with $J_i = J|_{V_i}$ and since J is invertible, each of the maps J_i is invertible. Since $J^* = -J$ and $J^* = \oplus J_i^*$ we see that each of the forms J_i is skew symmetric and invertible thus defines a symplectic space (V_i, J_i) over E_i. $\square$

Proof of Theorem 4.0.2. Let $g \in GL(V)$ be an element which is semi-simple with respect to the $Sp(V) \times Sp(V)$ action. The corresponding descendant is defined via the element $x = s(g) = \sigma(g)g$ which is semi-simple and belongs to $GL(V)^\sigma$. So let $x \in GL(V)^\sigma$ be a semi-simple element. We claim that (G_x, H_x) is a product of symmetric pairs as above. By the comment above

$$G_x = \prod_{i=1}^{n} GL(V_i, E_i)$$

Let now $h \in H = Sp(V, J)$ be an element in $H_x \subset G_x$. Since $h \in H$ we have $h^*J = Jh$. Since $h \in G_x$ we can write everything in blocks corresponding to the decompositions of V and V^*. Using the previous lemma we obtain that $h = diag(h_1, ..., h_r)$ with each $h_i : V_i \to V_i$ satisfy $h_i^* = J_i h_i J_i^{-1}$. Thus,

$$H_x = \prod_{i=1}^{n} Sp((V_i, J_i)/E_i)$$

This is exactly the claim.

$\square$

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