

CARATHÉODORY FUNCTIONS ON RIEMANN SURFACES AND REPRODUCING KERNEL SPACES

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ABSTRACT. Carathéodory functions, i.e. functions analytic in the open upper half-plane and with a positive real part there, play an important role in operator theory, $1D$ system theory and in the study of de Branges-Rovnyak spaces. The Herglotz integral representation theorem associates to each Carathéodory function a positive measure on the real line and hence allows to further examine these subjects. In this paper, we study these relations when the Riemann sphere is replaced by a real compact Riemann surface. The generalization of Herglotz's theorem to the compact real Riemann surface setting is presented. Furthermore, we study de Branges-Rovnyak spaces associated with functions with positive real-part defined on compact Riemann surfaces. Their elements are not anymore functions, but sections of a related line bundle.

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1. INTRODUCTION AND OVERVIEW

A Carathéodory function $\varphi(z)$, that is, analytic with positive real part in the open upper half-plane \mathbb{C}_+ , admits an integral representation,

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also known by the Herglotz's representation theorem (see e.g [10, 19]). More precisely, the Carathéodory function $\varphi(z)$ can be written as:

$$(1.1) \quad \varphi(z) = iA - iBz - i \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t),$$

where $A \in \mathbb{R}$, $B \geq 0$ and $d\mu(t)$ is a positive measure on the real line such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t^2+1} < \infty.$$

One of the main two objectives of this paper is to extend (1.1) to the non zero genus case; this is done in Theorem 3.3. The second point is to extend (1.2) also to the non zero genus case, and this result is presented in Theorem 4.1.

The right handside of (1.1) defines an analytic extension of $\varphi(z)$ to $\mathbb{C} \setminus \mathbb{R}$ such that $\overline{\varphi(\bar{z})} + \varphi(z) = 0$. Thus, for any $z, w \in \mathbb{C} \setminus \mathbb{R}$, we have

$$(1.2) \quad \frac{\varphi(z) + \overline{\varphi(w)}}{-i(z - \bar{w})} = B + \int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)(t-\bar{w})} = B + \left\langle \frac{1}{t-z}, \frac{1}{t-w} \right\rangle_{\mathbf{L}^2(d\mu)},$$

where $\mathbf{L}^2(d\mu)$ stands for the Lebesgue (Hilbert) space associated with the measure $d\mu$. Thus, the kernel $\frac{\varphi(z) + \overline{\varphi(w)}}{-i(z - \bar{w})}$ is positive in $\mathbb{C} \setminus \mathbb{R}$. When $B = 0$, the associated reproducing kernel Hilbert space, denoted by $\mathcal{L}(\varphi)$, is described in the theorem below.

Theorem 1.1 ([8, Section 5]). *The space $\mathcal{L}(\varphi)$ consists of the functions of the form*

$$(1.3) \quad F(z) = \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{t-z}$$

where $f \in \mathbf{L}^2(d\mu)$. Furthermore, $\mathcal{L}(\varphi)$ is invariant under the resolvent-like operators R_α , where for $\alpha \in \mathbb{C} \setminus \mathbb{R}$, is given by:

$$(1.4) \quad (R_\alpha F)(z) = \frac{F(z) - F(\alpha)}{z - \alpha}.$$

Finally, under the hypothesis $\int_{\mathbb{R}} d\mu(t) < \infty$, the elements of $\mathcal{L}(\varphi)$ satisfy:

$$\lim_{y \rightarrow \infty} F(iy) = 0.$$

Moreover, the resolvent operator satisfies $R_\alpha = (M - \alpha I)^{-1}$, where M is the multiplication operator defined by

$$M F(z) = zF(z) - \lim_{z \rightarrow \infty} zF(z).$$

We note that M corresponds, through (1.3), to the operator of multiplication by t in $\mathbf{L}^2(d\mu)$.

We provide here the outline of the proof in order to motivate the analysis presented in the sequel in the compact real Riemann setting.

Proof of Theorem 1.1: Let $N \in \mathbb{N}$, $w_1, \dots, w_N \in \mathbb{C} \setminus \mathbb{R}$ and $c_1 \cdots c_N \in \mathbb{C}$. Then,

$$F(z) \stackrel{\text{def}}{=} \sum_{j=1}^N c_j \frac{\varphi(z) + \overline{\varphi(w_j)}}{-i(z - \overline{w_j})} = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} f(t)$$

where

$$f(t) = \sum_{j=1}^N \frac{c_j}{t - \overline{w_j}} \in \mathbf{L}^2(d\mu).$$

In view of (1.2), we have

$$\|F\|_{\mathcal{L}(\varphi)}^2 = \|f\|_{\mathbf{L}^2(d\mu)}^2 = \sum_{\ell, j} \overline{c_\ell} \frac{\varphi(w_\ell) + \overline{\varphi(w_j)}}{-i(w_\ell - \overline{w_j})} c_j.$$

The first claim (Equation 1.3) follows by the fact that the linear span of the functions $\frac{1}{-i(z - \overline{w})}$, $w \in \mathbb{C} \setminus \mathbb{R}$ is dense in $\mathbf{L}^2(d\mu)$.

Next, let $F(z) = \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{t - z} \in \mathcal{L}(\varphi)$. Then,

$$(1.5) \quad (R_\alpha F)(z) = \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{(t - \alpha)(t - z)}$$

belongs to $\mathcal{L}(\varphi)$ since $f(t)/(t - \alpha) \in \mathbf{L}^2(d\mu)$ where α is lying outside the real line. \square

Furthermore, using (1.5), the structure identity

$$(1.6) \quad [R_\alpha f, g] - [f, R_\beta g] - (\alpha - \overline{\beta})[R_\alpha f, R_\beta g] = 0, \quad \alpha, \beta \in \mathbb{C} \setminus \mathbb{R},$$

holds in the $\mathcal{L}(\varphi)$ spaces. In fact, this is an "if and only if" relation. If the identity (1.6) holds in the space \mathcal{L} of functions analytic in $\mathbb{C} \setminus \mathbb{R}$, then $\mathcal{L} = \mathcal{L}(\varphi)$ for some Carathéodory function $\varphi(z)$ (see [8, Theorem 6]).

Using the observation that an $\mathbf{L}^2(d\mu)$ space is finite dimensional if and only if the measure $d\mu$ is has only a singular part, consisting of a finite number of jumps, we may continue and mention the following result (see for instance [9]):

Theorem 1.2. *Let $\varphi(z)$ be a Carathéodory function associated via (1.1) to a positive measure $d\mu$ and let $\mathcal{L}(\varphi)$ be the corresponding reproducing kernel Hilbert space. Then the following are equivalent:*

- (1) $\mathcal{L}(\varphi)$ is finite dimensional.
- (2) $\dim \mathbf{L}^2(d\mu) < \infty$.

- (3) $d\mu$ is a jump measure with a finite number of jumps.
 (4) The Carathéodory function is of the form

$$\varphi(z) = iA + iBz + \sum_{j=1}^N \frac{ic_j}{z - t_j},$$

where $c_j, B > 0$, $A, t_j \in \mathbb{R}$ for all $1 \leq j \leq N$.

Remark 1.3. There are two different ways to obtain the positive measure $d\mu$ given in (1.1).

- (1) Using the Cauchy formula on the boundary and the Banach-Alaoglu Theorem.
 (2) Using the spectral theorem for R_0 in the space $\mathcal{L}(\varphi)$. In this case, R_0 is self-adjoint and the measure $d\mu$ is given by $d\mu(t) = \langle dE(t)u, u \rangle$ where E is the spectral measure of R_0 .

In this paper we focus on the first approach, while in [3] we explore the second approach.

We mention here that Carathéodory functions are the characteristic functions or transfer functions of selfadjoint vessel or impedance $2D$ systems, respectively. Furthermore, they are also related to de Branges-Rovnyak spaces $\mathcal{L}(\varphi)$ of sections of certain vector bundles defined on compact Riemann surfaces of non zero genus. These subjects and interconnections are further studied by the authors in [3].

Outline of the paper: The paper consists of five sections besides the introduction. In Section 2, we give a brief overview of compact real Riemann surfaces and the associated Cauchy kernels.

In Section 3, we describe explicitly the Green function on X in terms of the canonical homology basis. As a consequence, we present the Herglotz representation theorem for compact real Riemann surfaces. We utilize the integral representation of Carathéodory functions in order to study, in Section 4, the de Branges space $\mathcal{L}(\varphi)$.

In Section 5, we examine the case where $\varphi(z)$ is a single-valued function which defines a contractive function $s(z)$ through the Cayley transformation. Hence, we may determine the relation between the de Branges spaces $\mathcal{L}(\varphi)$ and the de Branges Rovnyak space $\mathcal{H}(s)$ associated to s . Finally, in Section 6, we summarize some of the results by comparing the $\mathcal{L}(\varphi)$ theory in the genus zero case and the real compact Riemann surfaces of genus $g > 0$.

2. PRELIMINARIES

In this section, we give a brief review of the basic properties and definitions of compact real Riemann surfaces. We replace the open upper-half plane (or, more precisely, its double, i.e the Riemann sphere) by a compact real Riemann surface X of genus $g > 0$.

A survey of the main tools required in the present study (including the prime form and the Jacobian) can be found in [6, Section 2], and in particular the descriptions of the Jacobian variety of a real curve and the real torii is in [22]. For general background, we refer to [12, 14, 16, 17] and [18].

It is crucial to choose a canonical basis to the homology group $H_1(X, \mathbb{Z})$ which is symmetric, in some sense, under the involution τ (for more details we refer to [15]). Here we use the conventions as in [6, 22]). Let $X_{\mathbb{R}}$ be the set of the invariant points under τ , $X_{\mathbb{R}} = \{p \in X \mid \tau(p) = p\}$, which is always assumed to be not empty. Then $X_{\mathbb{R}}$ consists of k connected components denoted by X_j where $j = 0, \dots, k-1$ (disjoint analytic simple closed curves). We choose for each component X_j a point $p_j \in X_j$. Then, we set $A_{g+1-k+j} = X_j$ and $B_{g+1-k+j} = C_j - \tau(C_j)$, where $j = 1, \dots, k-1$ and C_j is a path from p_0 to p_j which does not contain any other fixed point. We can extend to the homology basis $A_1, \dots, A_g, B_1, \dots, B_g$ under which the involution is given by $\begin{pmatrix} I & H \\ 0 & -I \end{pmatrix}$, where the matrix H is given by

$$H = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

for the dividing case and the non-dividing case, respectively. In both cases, H is of rank of $g + 1 - k$. Then, we choose a normalized basis of holomorphic differentials on X satisfying $\int_{A_i} \omega_j = \delta_{ij}$. The matrix $Z \in \mathbb{C}^{g \times g}$, with entries $Z_{i,j} = \int_{B_i} \omega_j$, is symmetric, with positive real part, satisfies

$$Z^* = H - Z$$

and is referred as the period matrix of X associated with the basis $(\omega_j)_{j=1}^g$. The Jacobian variety is defined by $J(X) = \mathbb{C}^g \backslash \Gamma$, where $\Gamma = \mathbb{Z}^g + ZZ^g$, and the Abel-Jacobi map from X to the Jacobian variety is

given by

$$\mu : p \rightarrow \begin{pmatrix} \int_{p_0}^p \omega_1 \\ \vdots \\ \int_{p_0}^p \omega_g \end{pmatrix}.$$

It is convenient to define

$$(2.1) \quad Z = \frac{1}{2}H + iY^{-1}.$$

We denote the universal covering of X by $\pi : \widetilde{X} \rightarrow X$. The group of deck transformations of X , denoted by the $\text{Deck}(\widetilde{X}/X)$, consists of the homeomorphisms $\mathcal{T} : \widetilde{X} \rightarrow \widetilde{X}$ such that $\pi_X \circ \mathcal{T} = \pi_X$. It is well-known that the group of deck transformations on the universal covering is isomorphic to the fundamental group $\pi_1(X)$.

The analogue of the kernel $\frac{1}{-i(z-\bar{w})}$, is given by $\frac{K_\zeta(u, \tau(v))}{-i}$ where

$$K_\zeta(u, v) \stackrel{\text{def}}{=} \frac{\vartheta[\zeta](v-u)}{\vartheta[\zeta](0)E(v, u)}.$$

The analogue of the kernel $\frac{1-s(z)\overline{s(w)}}{-i(z-\bar{w})}$ is now given by the expression

$$K_{\tilde{\zeta}, s}(u, v) = \frac{\vartheta[\tilde{\zeta}](\tau(v)-u)}{i\vartheta[\tilde{\zeta}](0)E(u, \tau(v))} - s(u) \frac{\vartheta[\zeta](\tau(v)-u)}{i\vartheta[\zeta](0)E(u, \tau(v))} \overline{s(v)},$$

where u and v are points on X (see [21] and [22]). Furthermore, ζ and $\tilde{\zeta}$ are points on the Jacobian $J(X)$ (in fact ζ and $\tilde{\zeta}$ belong to the real torii T_ν , see [23]) of X such that $\vartheta(\zeta)$ and $\vartheta(\tilde{\zeta})$ are nonzero and:

- (1) $\vartheta[\zeta]$ denotes the theta function of X with characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ where $\zeta = b + Za$ (with a and b in \mathbb{R}^g).
- (2) $E(u, v)$ is the prime form on X , for more details see [12, 18].
- (3) For fixed v , the map $u \mapsto K_{\tilde{\zeta}, s}(u, v)$ is a multiplicative half order differential (with multipliers corresponding to $\tilde{\zeta}$).
- (4) s is a map of line bundles on X with multipliers corresponding to $\tilde{\zeta} - \zeta$ and satisfying $s(u)s(\tau(u))^* = 1$.

The analogue of the operators (1.4) is given now by

$$R_\alpha^y f(u) = \frac{f(u)}{y(u) - \alpha} - \sum_{j=1}^n \frac{1}{dy(u^{(j)})} \frac{\vartheta[\zeta](u^{(j)} - u)}{\vartheta[\zeta](0)E(u^{(j)}, u)} f(u^{(j)}),$$

where y is a real meromorphic function of degree n and $\alpha \in \mathbb{C}$ is such that there are n distinct points $u^{(j)}$ in X such that $y(u^{(j)}) = \alpha$ and

where f is a section of $L_\zeta \otimes \Delta$ analytic at the points $u^{(j)}$. Furthermore, ([6, Lemma 4.3]) the Cauchy kernels are eigenvectors of R_α^y with eigenvalues $\frac{1}{y(w)-\alpha}$.

We conclude with the definition of the model operator, M^y [23, Equation 3-3], satisfying $(M^y - \alpha I)^{-1} = R_\alpha^y$ for α large enough. It is defined on sections of the line bundle $L_\zeta \otimes \Delta$ analytic at the neighborhood of the poles of y and is explicitly given by

$$(2.2) \quad M^y f(u) = y(u)f(u) + \sum_{m=1}^n c_m f(p^{(m)}) \frac{\vartheta[\zeta](p^{(m)} - u)}{\vartheta[\zeta](0)E(p^{(m)}, u)},$$

where $y(u)$ is a meromorphic function on X with n distinct simple poles, $p^{(1)}, \dots, p^{(n)}$.

3. HERGLOTZ THEOREM FOR COMPACT REAL RIEMANN SURFACES

We first develop the analogue of Herglotz's formula for analytic functions with a positive real part in X_+ , instead of \mathbb{C}_+ . We consider the case of multi-valued functions but with purely imaginary period, i.e. multi-valued functions that satisfy

$$\varphi(\mathcal{T}(\tilde{p})) = \varphi(\tilde{p}) + \chi(\mathcal{T}).$$

Here \mathcal{T} is an element in the group of deck transformations on the universal covering of X and

$$\chi : \pi_1(X) \rightarrow i\mathbb{R},$$

is a homomorphism of groups. We call such a mapping an *additive function*. Although in general it is not uniquely defined, the real part of $\varphi(p)$ is well-defined.

The involution τ is extended on the universal covering of X . In particular, for $\tilde{x} \in \tilde{X}$, an inverse under π of an element in $X_{\mathbb{R}}$, there exists $\mathcal{T}_{\tilde{x}} \in \text{Deck}(\tilde{X}/X)$ such that $\tau(\tilde{x}) = \mathcal{T}_{\tilde{x}}(\tilde{x})$. Hence, since $\text{Deck}(\tilde{X}/X)$ is isomorphic to $\pi_1(X)$, we write $\mathcal{T}_{\tilde{x}}$ in the form $\mathcal{T}_{\tilde{x}} = \sum_{j=1}^g m_j A_j + n_j B_j$, and we extensively use the notation

$$\begin{aligned} n(\cdot) : \tilde{X} &\longrightarrow \mathbb{Z}^g \\ \tilde{x} &\longrightarrow (n_1 \cdots n_g)^t. \end{aligned}$$

We note that when $\tilde{x} \in \tilde{X}_0$ it follows that $n(\tilde{x}) = 0$ and $n(\tilde{x}) = e_{g+1-k+j}$ whenever $\tilde{x} \in \tilde{X}_j$ for $j = 1, \dots, k-1$ (where the set e_1, \dots, e_g forms the canonical basis of \mathbb{R}^g).

Theorem 3.1. *Let X be a compact real Riemann surface and let $\psi(p)$ be a positive harmonic function defined on $X \setminus X_{\mathbb{R}}$. Then for every $p \in X \setminus X_{\mathbb{R}}$ there exists a positive measure $d\eta(p, x)$ on $X_{\mathbb{R}}$ such that*

$$(3.1) \quad \psi(p) = \int_{X_{\mathbb{R}}} \psi(x) d\eta(p, x).$$

We start by presenting a preliminary lemma, revealing a useful property of the prime form.

Lemma 3.2. *Let x be an element of X_j . Then the prime form satisfies the following relation*

$$(3.2) \quad \begin{aligned} \overline{\frac{\partial}{\partial x} \ln E(\tau(p), x)} &= \frac{\partial}{\partial x} \ln E(p, x) - 2\pi i \omega_{g-k-1+j}(x) \\ &= \frac{\partial}{\partial x} \ln E(p, x) - 2\pi i \left[\omega_1(x) \cdots \omega_g(x) \right] n(x). \end{aligned}$$

Proof: Let $x \in X_j$, where $j = 1, 2, \dots, k-1$. Then, τ is lifted to the universal covering as follows

$$\tilde{x} - \tau(\tilde{x}) = \sum_{\ell=1}^g m_{\ell} A_{\ell} + n_{\ell} B_{\ell},$$

where $n_{\ell} = \delta(g - k - 1 + j - \ell)$ and where δ stands for the Kronecker delta. We also recall that the prime form (see [6, Lemma 2.3]) satisfies

$$(3.3) \quad \begin{aligned} E(\tilde{p}, \tilde{u}_1) &= E(\tilde{p}, \tilde{u}_2) \exp \left(-i\pi n^t \Gamma n + 2\pi i (\tilde{\mu}(\tilde{p}) - \tilde{\mu}(\tilde{u}_2))^t n \right) \times \\ &\times \exp \left(2\pi i (\beta_0^t n - \alpha_0^t m) \right), \end{aligned}$$

where $\tilde{\mu}$ is the lifting of the Abel-Jacobi mapping to the universal covering and where $\zeta = \alpha_0 + \beta_0 \Gamma$. Thus, choosing $\tau(\tilde{u}_2) = \tilde{u}_1 = \tilde{x}$, the relation in (3.3) becomes

$$(3.4) \quad \begin{aligned} \ln E(\tilde{p}, \tau(\tilde{x})) &= \ln E(\tilde{p}, \tilde{x}) - i\pi \Gamma_{jj} + 2\pi i (\tilde{\mu}(\tilde{p}) - \tilde{\mu}(\tilde{x}))_j + \\ &+ 2\pi i (\beta_0^t n - \alpha_0^t m). \end{aligned}$$

We note that by using [6, Lemma 2.4], the prime form satisfies the identity $\overline{E(\tau(p), x)} = E(p, \tau(x))$. It remains to differentiate (3.4) with respect to x and (3.2) follows. \square

Proof of Theorem 3.1: We show that the expression

$$(3.5) \quad \begin{aligned} G(p, x) &\stackrel{\text{def}}{=} \pi \left[\omega_1(x) \cdots \omega_g(x) \right] \cdot \left(\frac{n(x)}{2} + i(Yp) \right) - \\ &- \frac{i}{2} \frac{\partial}{\partial x} \ln E(p, x), \end{aligned}$$

is the differential with respect to x of the Green function, where $x \in X_{\mathbb{R}}$, $p \in X \setminus X_{\mathbb{R}}$ and where $\omega(x)$ is a section of the canonical bundle (denoted by K_X and for an atlas (V_j, z_j) defining the analytic structure of X , is given by cocycles dz_j/dz_i). Here and in the following pages, with an abuse of notation, Yp denotes $Y\tilde{\mu}(\tilde{p})$. The existence of a Green function on a Riemann surface is a well-known result, see for instance [7, Chapter V] or [20, Chapter X]. Therefore, there exists a (unique) Green function, denoted by $g(p, x)$, with the differential $G(p, x)$ which contains singularities of the form $\frac{1}{x-p}$ along its diagonal. Hence, it is enough to show that the expression in (3.5) and the Green function satisfy the upcoming properties:

- (1) *The function $g(x, p)$ contains a logarithmic singularity while $G(x, p)$ has a simple pole at $p = x$.* It follows immediately by using the prime form properties and moving to local coordinates that the following relation holds (see for instance [12, Section II]):

$$\frac{i}{2} \frac{\partial}{\partial x} \ln E(p, x) = \frac{i}{2} \frac{\partial}{\partial v} \ln(t(u) - t(v)) = \frac{i}{2(t(u) - t(v))}.$$

- (2) *The real part of the differential $G(x, p)$ is single-valued:* Let p and p_1 be two elements of \tilde{X} which are the pre-images of the same element in X_j , i.e. $\pi(p) = \pi(p_1) \in X_j$. It follows, using (2.1), that

$$(3.6) \quad \tilde{\mu}(p) - \tilde{\mu}(p_1) = n + \Gamma m = n + \left(\frac{1}{2}H + iY^{-1} \right) m,$$

for some $n, m \in \mathbb{R}^g$ and thus, using again [6, Lemma 2.3] we have, modulo $2\pi i$:

$$\begin{aligned} \ln(E(p_1, x)) &= \ln \left(E(p, x) \exp \left(2\pi i (\mu(x) - \mu(p))^t m \right) \times \right. \\ &\quad \left. \exp \left(2\pi i (\beta_0^t m - \alpha_0^t n) - i\pi m^t \Gamma m \right) \right) \\ &= \ln E(p, x) - \frac{i}{2} \pi m^t H m - \pi m^t Y^{-1} m + \\ &\quad 2\pi i (\tilde{\mu}(x) - \tilde{\mu}(p))^t m + 2\pi i (\beta_0^t m - \alpha_0^t n). \end{aligned}$$

Then, the real part of a multiplier of $\ln(E(p, x))$ is:

$$(3.7) \quad \Re \left(\ln E(p, x) - \ln E(p_1, x) \right) = 2\pi \left(\frac{1}{2} m^t Y^{-1} + \Im (\mu(p) - \mu(x))^t \right) m.$$

Clearly, using (3.6), we have that

$$m = Y \Im (\tilde{\mu}(p) - \tilde{\mu}(p_1))$$

and hence the derivative with respect to x of (3.7), is

$$\begin{aligned} \frac{\partial}{\partial x} \Re (\ln (E(p, x)) - \ln (E(p_1, x))) \\ &= -2\pi \Im \left[\omega_1(x) \cdots \omega_g(x) \right] m \\ &= -2\pi \Im \left[\omega_1(x) \cdots \omega_g(x) \right] Y \tilde{\mu}(p - p_1). \end{aligned}$$

Hence, $G(x, p)$ has the appropriate singularity and has a single-valued real-part if it is of the following form:

$$\frac{\partial}{\partial x} \ln (E(p, x)) + 2\pi \left[\omega_1(x) \cdots \omega_g(x) \right] Y \tilde{\mu}(p) + h(x),$$

for some $h(x)$ with purely imaginary periods.

- (3) *The real part of the complex Green function vanishes on the boundary components:* Let $x \in X_j$ for some $0 \leq j \leq k-1$ and let $p \in X_l$ for some $0 \leq l \leq k-1$ such that $p \neq x$. Then, we integrate $G(x, p)$ with respect to x and note that the integration of the vector $\left[\omega_1(x) \cdots \omega_g(x) \right]$ is just the Abel-Jacobi mapping at x . Then, the Green function is:

$$g(x, p) = \left(\frac{n(x)}{2} + i(Yp) \right) \mu(x) - \frac{i}{2} \ln E(p, x).$$

We use the equality, see [6, Lemma 2.4],

$$E(x, p) = \overline{E(\tau(x), \tau(p))}$$

to conclude that whenever x and p are both real, the relation

$$\begin{aligned} \overline{\frac{\partial}{\partial x} \ln (E(x, p))} &= \frac{\partial}{\partial x} \ln (E(\tau(x), \tau(p))) \\ &= \frac{\partial}{\partial x} \ln (E(x, p)) + 2\pi i \omega_{g-k+j-1}(x) \end{aligned}$$

holds. Hence, the real part of $g(x, p)$, using Equation 3.2, is equal to

$$\begin{aligned} \Re(g(x, p)) &= i(Yp)\mu(x) - \frac{i}{2} \ln E(p, x) + \overline{i(Yp)\mu(x)} - \\ &\quad \overline{\frac{i}{2} \ln E(p, x) + \Re(h(x))} \\ &= \Re \frac{i}{2} (\ln E(\tau(p), \tau(x)) - \ln E(p, x)) + \Re(h(x)) \\ &= \frac{\pi}{2} \omega(x)n(x) + \Re(h(x)), \end{aligned}$$

and therefore, setting $\Re h(x) = -\frac{\pi}{2} \omega(x)n(x)$, the Green function vanishes on the real points.

Thus, $G(x, p)$ is the differential of the complex Green function and so, for any $p \in X \setminus X_{\mathbb{R}}$ and for sufficient small ε , defines the solution to the Dirichlet problem, i.e.

$$\psi(p) = \int_{X_{\mathbb{R}}(\varepsilon)} \psi(x)G(x, p).$$

Here, the integration contour is a collection of smooth simple closed curves located within a distance ε approximating $X_{\mathbb{R}}$. We then consider a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then, by the Banach-Alaoglu Theorem (see for instance, [13, p. 223]), there exists a subsequence $(\varepsilon_{n_k})_{k \in \mathbb{N}}$ such that the limit

$$\lim_{k \rightarrow \infty} \int_{X_{\mathbb{R}}(\varepsilon_{n_k})} \psi(x)G(x, p)$$

exists. Thus, the weak-star limit defines a positive measure on $X_{\mathbb{R}}$ satisfying (3.1). \square

Using the previous result, we may state the Herglotz theorem for real compact Riemann surfaces.

Theorem 3.3. *Let X be a compact real Riemann surface of dividing-type. Then an additive function $\varphi(x)$ analytic in $X \setminus X_{\mathbb{R}}$ with positive real part in $X \setminus X_{\mathbb{R}}$ and, furthermore, satisfies*

$$\varphi(p) + \overline{\varphi(\tau(p))} = 0, \quad p \in X \setminus X_{\mathbb{R}},$$

if and only if

$$\begin{aligned} \varphi(p) &= \frac{\pi}{2} \int_{X_{\mathbb{R}}} \left[\omega_1(x) \cdots \omega_g(x) \right] n(\tilde{x}) \frac{d\eta(x)}{\omega(x)} - \frac{i}{2} \int_{X_{\mathbb{R}}} \frac{\partial}{\partial x} \ln E(p, x) \frac{d\eta(x)}{\omega(x)} + \\ (3.8) \quad &\pi i \int_{X_{\mathbb{R}}} \left[\omega_1(x) \cdots \omega_g(x) \right] (Yp) \frac{d\eta(x)}{\omega(x)} + iM. \end{aligned}$$

Here, M is a real number, $d\eta$ is a positive finite measure on $X_{\mathbb{R}}$, $\omega(x)$ is a section of the canonical line bundle which is positive with respect to the measure $d\eta$.

Proof: We start with the "if" part as we compute $\overline{\varphi(\tau(p))}$:

$$\begin{aligned} \overline{\varphi(\tau(p))} &= \frac{\pi}{2} \int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] n(\tilde{x}) \frac{d\eta(x)}{\omega(x)} - \\ &\quad \pi i \int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] (\overline{Yp}) \frac{d\eta(x)}{\omega(x)} + \\ &\quad \frac{i}{2} \int_{X_{\mathbb{R}}} \frac{\partial}{\partial x} \ln \overline{E(\tau(p), x)} \frac{d\eta(x)}{\omega(x)} - iM. \end{aligned}$$

Thus, using Lemma 3.2 and since ω is real (i.e. $\overline{\tau(\omega_i)} = \omega_i$), we have:

$$\begin{aligned} \overline{\varphi(\tau(p))} &= \frac{\pi}{2} \int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] n(\tilde{x}) \frac{d\eta(x)}{\omega(x)} - \\ &\quad \pi i \int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] (Yp) \frac{d\eta(x)}{\omega(x)} + \\ (3.9) \quad &\quad \frac{i}{2} \int_{X_{\mathbb{R}}} \frac{\partial}{\partial x} \ln E(p, x) \frac{d\eta(x)}{\omega(x)} - \pi \sum_{j=1}^{k-1} \int_{X_j} \omega_j(x) \frac{d\eta(x)}{\omega(x)} - iM. \end{aligned}$$

Summing up (3.8) and (3.9), leads to

$$\begin{aligned} \varphi(p) + \overline{\varphi(\tau(p))} &= \pi \int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] n(\tilde{x}) \frac{d\eta(x)}{\omega(x)} - \\ &\quad \pi \sum_{j=1}^{k-1} \int_{X_j} \omega_j(x) \frac{d\eta(x)}{\omega(x)} = 0. \end{aligned}$$

For the "only if" statement: The real part of $\varphi(p)$ is positive, harmonic and with a single-valued real part in $X \setminus X_{\mathbb{R}}$. Thus, by Theorem 3.1, $\Re \varphi(p)$ has an integral representation as given in (3.1) for some positive measure $d\eta_{\varphi}$ on $X_{\mathbb{R}}$. Finally, it is well-known that two analytic functions defined on a connected domain with the same real part differ only by some imaginary constant. Hence we may summarize that

$$\varphi(p) = \int_{X_{\mathbb{R}}} G(p, x) d\nu_{\varphi}(x) + iM,$$

for some $M \in \mathbb{R}$. □

In the case where $X = \mathbb{P}^1$ coupled with the anti-holomorphic involution $z \rightarrow \bar{z}$, we set $\omega = \frac{dt}{t^2+1}$ and then (1.1) can be extracted from (3.8) by

setting,

$$\begin{aligned} d\nu(t) &= \frac{1}{2}d\eta(t)(t^2 + 1), & B &= \frac{1}{2}\eta(\infty), \\ A &= M - \frac{1}{2}\int_I t d\eta(t) + \frac{1}{2}\int_{\mathbb{R}\setminus I} \frac{d\eta(t)}{t}, \end{aligned}$$

where I is any interval of \mathbb{R} containing zero.

Similarly, in the case of the torus, one may deduce H. Villat's formula, see [11]. (Akhiezer in [1, Section 56] presented a different but equivalent formula).

4. DE BRANGES $\mathcal{L}(\varphi)$ SPACES IN THE NONZERO GENUS CASE

In this section, we further study the reproducing kernel Hilbert space associated with an additive function defined on a real compact Riemann space. To do so, we utilize the Herglotz's integral representation proved in the previous section in order to examine $\mathcal{L}(\varphi)$ spaces and their properties. First, we introduce the analogue of formula (1.2).

Theorem 4.1. *Let X be a compact real Riemann surface of dividing type, $\zeta \in T_0$ and let φ be an analytic with positive real part in X_+ . Then, the identity*

$$\begin{aligned} &\int_{X_{\mathbb{R}}} \frac{\vartheta[\zeta](p-x)}{\vartheta[\zeta](0)E(x,p)} \frac{\vartheta[\zeta](x-\tau(q))}{\vartheta[\zeta](0)E(x,\tau(q))} \frac{d\eta(x)}{\omega(x)} = \frac{\vartheta[\zeta](p-\tau(q))}{\vartheta[\zeta](0)E(\tau(q),p)} \times \\ &\quad \times \left[\left(\varphi(p) + \overline{\varphi(\tau(q))} \right) + \sum_{j=0}^{k-1} a_{jj} \frac{\partial}{\partial z_j} \ln \frac{\vartheta(\zeta)}{\vartheta(\zeta + p - \tau(q))} - \right. \\ (4.1) \quad &\left. - 2\pi i \operatorname{Row}_{i=0,\dots,g} \left(\sum_{j=0}^{k-1} a_{ji} \right) Y(p - \tau(q)), \right. \end{aligned}$$

holds where

$$(4.2) \quad a_{ji} \stackrel{\text{def}}{=} \int_{X_j} \frac{\omega_i(x)}{\omega(x)} d\eta(x), \quad j = 0, \dots, k-1, \quad i = 1, \dots, g.$$

Before heading to prove Theorem 4.1, we make a number of remarks. The left hand side of (4.1) may be written as

$$\langle K_{\zeta}(p, x), K_{\zeta}(x, \tau(q)) \rangle_{\mathbf{L}^2(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{\omega(x)})}$$

and hence, it is precise the counterpart of the right hand side of (1.2). Furthermore, whenever we additionally assume zero cycles along the boundary components, that is,

$$(4.3) \quad \int_{X_j} \frac{\omega_i(x)}{\omega(x)} d\eta(x) = 0, \quad j = 0, \dots, k-1,$$

the right hand side of (4.1) is the counterpart of the left hand side of (1.2). Hence, we may summarize and present the following result.

Corollary 4.2. *Let X be a compact real Riemann surface of dividing type, let $\zeta \in T_0$ and let φ be an additive function on X such that (4.3) holds. Then, the identity*

$$(4.4) \quad (\varphi(p) + \overline{\varphi(\tau(q))}) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} = \langle K_\zeta(p, u), K_\zeta(u, \tau(q)) \rangle_{\mathbf{L}^2(X_{\mathbb{R}}, L_\zeta \otimes \Delta, \frac{d\eta(x)}{w(x)})},$$

holds.

Proof of Theorem 4.1: Using the Herglotz-type formula (3.8), we may write

$$(4.5) \quad (\varphi(p) + \overline{\varphi(\tau(q))}) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} = 2\pi i \left(\int_{X_{\mathbb{R}}} [\omega_1(x) \cdots \omega_g(x)] Y(p - \tau(q)) \frac{d\eta(x)}{\omega(x)} \right) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} - i \int_{X_{\mathbb{R}}} \left(\frac{\partial}{\partial x} \ln E(p, x) - \frac{\partial}{\partial x} \ln E(\tau(q), x) \right) \frac{d\eta(x)}{\omega(x)} \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)}.$$

Furthermore, by [12, Proposition 2.10, p. 25], we have

$$(4.6) \quad \frac{\partial}{\partial x} \ln \frac{E(p, x)}{E(\tau(q), x)} + \sum_{j=1}^g \left(\frac{\partial}{\partial z_j} \ln \vartheta(\zeta + p - \tau(q)) - \frac{\partial}{\partial z_j} \ln \vartheta(\zeta) \right) \omega_j(x) = \frac{E(\tau(q), p)}{E(x, \tau(q))E(x, p)} \frac{\vartheta[\zeta](x - \tau(q))\vartheta[\zeta](p - x)}{\vartheta[\zeta](p - \tau(q))\vartheta[\zeta](0)}.$$

Thus, multiplying both sides of (4.6) by $\frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)}$, leads to

$$(4.7) \quad \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} \frac{\partial}{\partial x} \ln \frac{E(p, x)}{E(\tau(q), x)} = \frac{\vartheta[\zeta](x - \tau(q))}{\vartheta[\zeta](0)E(x, \tau(q))} \frac{\vartheta[\zeta](p - x)}{\vartheta[\zeta](0)E(x, p)} - \sum_{j=1}^g \frac{\partial}{\partial z_j} (\ln \vartheta(\zeta + p - \tau(q)) - \ln \vartheta(\zeta)) \omega_j(x) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)}.$$

Finally, by substituting (4.7) into (4.5), we conclude that the identity

$$\begin{aligned}
 (\varphi(p) - \overline{\varphi(\tau(q))}) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} &= \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)} \times \\
 &\left[\frac{i}{2} \sum_{j=1}^g \int_{X_j} \frac{\omega_j(x) d\eta(x)}{\omega(x)} \frac{\partial}{\partial z_j} (\ln \vartheta(\zeta + p - \tau(q)) - \ln \vartheta(\zeta)) + \right. \\
 &2\pi i \int_{X_{\mathbb{R}}} \frac{[\omega_1(x) \cdots \omega_g(x)]}{\omega(x)} Y(p - \tau(q)) d\eta(x) \left. - \right. \\
 &\left. \frac{i}{2} \int_{X_{\mathbb{R}}} \frac{\vartheta[\zeta](p - x)}{\vartheta[\zeta](0)E(x, p)} \frac{\vartheta[\zeta](x - \tau(q))}{\vartheta[\zeta](0)E(x, \tau(q))} \frac{d\eta(x)}{\omega(x)} \right]
 \end{aligned}$$

follows. Setting a_j as in (4.2), completes the proof. \square

From this point and onward we assume that (4.3) holds.

Definition 4.3. *Let $\varphi(x)$ be analytic in $X \setminus X_{\mathbb{R}}$ with positive real part in $X \setminus X_{\mathbb{R}}$. The reproducing kernel Hilbert space of sections of the line bundle $L_{\zeta} \otimes \Delta$ with the reproducing kernel*

$$K(p, q) = (\varphi(p) + \overline{\varphi(\tau(q))}) \frac{\vartheta[\zeta](p - \tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)},$$

is denoted by $\mathcal{L}(\varphi)$.

The analogue of the first part of Theorem 1.1 is given below in Theorem 4.5. However, we first present a preliminary lemma that is required during this section (see [2, Ex. 6.3.2], in the unit-disk case).

Lemma 4.4. *Let X be a compact real Riemann surface of dividing type. Then the linear span of Cauchy kernels $\frac{\vartheta[\zeta](x-u)}{i\vartheta[\zeta](0)E(u, x)}$ where u varies in $X \setminus X_{\mathbb{R}}$ is dense in $\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right)$.*

Proof: Let us assume that a section f of $L_{\zeta} \otimes \Delta$ satisfies

$$(4.8) \quad \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) f(x) \frac{d\eta(x)}{\omega(x)} = 0,$$

for all $u \in X \setminus X_{\mathbb{R}}$. We recall that by [5], there exists an isometric isomorphism from $\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right)$ to $(\mathbf{L}^2(\mathbb{T}))^n$ and therefore there exists an orthogonal decomposition, see [5, Equation 4.14],

$$\begin{aligned}
 &\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right) \\
 &= \mathbf{H}^2\left(X_+, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right) \oplus \mathbf{H}^2\left(X_-, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right).
 \end{aligned}$$

Furthermore, Equation 4.8, for $u \in X_+$ is just the projection from $\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right)$ into $\mathbf{H}^2\left(X_+, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right)$. Thus, $P_+(f)(u) = 0$ and, similarly, $P_-(f)(u) = 0$ and we may conclude that $f = 0$ and the claim follows. \square

Theorem 4.5. *The elements of $\mathcal{L}(\varphi)$ are of the form*

$$(4.9) \quad F(u) = \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) f(x) \frac{d\eta(x)}{w(x)},$$

where $f(x)$ is a section of $L_{\zeta} \otimes \Delta$ which is square summable with respect to $\frac{d\eta(x)}{w(x)}$.

Proof: Equation 4.9 follows by Corollary 4.2. Let us set $N \in \mathbb{N}$, then for any choice of $w_1, \dots, w_N \in X \setminus X_{\mathbb{R}}$ and $c_1 \cdots c_N \in \mathbb{C}$, the identity

$$(4.10) \quad \begin{aligned} F(u) &\stackrel{\text{def}}{=} \sum_{j=1}^n c_j (\varphi(u) + \overline{\varphi(w_j)}) K_{\zeta}(u, w_j) \\ &= \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) f(x) \frac{d\eta(x)}{w(x)} \end{aligned}$$

holds, where

$$f(u) = \sum_{j=1}^n c_j K_{\zeta}(w_j, u) \in \mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right).$$

Due to Lemma 4.4, the linear span of the kernels (4.4) is dense in $\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(x)}{w(x)}\right)$ and hence (4.9) follows. \square

Theorem 4.6. *The norm of an element F in $\mathcal{L}(\varphi)$ is given by*

$$\|F\|_{\mathcal{L}(\varphi)} \stackrel{\text{def}}{=} \|f\|_{\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(t)}{w(t)}\right)}.$$

Proof: Since, by Lemma 4.4, the linear span of the kernels (4.4) is dense in $\mathbf{L}^2(d\eta)$, it is enough to check the equality of the norms for a linear combination of the Cauchy kernels. The norm of an element in the reproducing kernel Hilbert space $\mathcal{L}(\varphi)$ is given by

$$\|F\|_{\mathcal{L}(\varphi)}^2 = \sum_{\ell, j=1}^n \overline{c_{\ell}} (\varphi(u_{\ell}) + \overline{\varphi(u_j)}) K_{\zeta}(u_{\ell}, u_j) c_j.$$

Then, by (4.10)

$$\|F\|_{\mathcal{L}(\varphi)}^2 = \sum_{\ell, j=1}^n \overline{c_j} \langle K_{\zeta}(u_{\ell}, w), K_{\zeta}(w, u_j) \rangle_{\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(t)}{w(t)}\right)} c_{\ell},$$

which is exactly the norm of $f(w)$ in $\mathbf{L}^2\left(X_{\mathbb{R}}, L_{\zeta} \otimes \Delta, \frac{d\eta(t)}{w(t)}\right)$. \square

As an immediate consequence, whenever y is a real function, we may state an additional result. It follows that M^y is simply the multiplication operator in $\mathbf{L}^2(d\eta)$.

Theorem 4.7. *Let y be a meromorphic function with simple poles such that the poles of y do not belong to the support of the measure $d\eta$. Then, the multiplication model operator M^y , defined on $\mathcal{L}(\varphi)$, satisfies the following properties:*

(1) M^y is given explicitly by

$$(4.11) \quad (M^y F)(u) = \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) f(x) y(x) \frac{d\eta(x)}{\omega(x)},$$

where f is a section of $L_{\zeta} \otimes \Delta$, which is square summable with respect to $d\eta$.

(2) $\mathcal{L}(\varphi)$ is invariant under M^y .

(3) M^y is bounded.

Proof: Considering the model operator (2.2) together with Theorem 4.5, we conclude the following:

$$\begin{aligned} (M^y F)(u) &= y(u)F(u) + \sum_{m=1}^n c_m F(p_m) K_{\zeta}(u, p_m) \\ &= y(u)F(u) + \sum_{m=1}^n c_m \int_{X_{\mathbb{R}}} f(x) K_{\zeta}(p_m, x) \frac{d\eta(x)}{\omega(x)} K_{\zeta}(u, p_m) \\ &= y(u)F(u) + \int_{X_{\mathbb{R}}} f(x) \frac{d\eta(x)}{\omega(x)} \sum_{m=1}^n c_m K_{\zeta}(p_m, x) K_{\zeta}(u, p_m), \end{aligned}$$

where $p^{(1)}, \dots, p^{(n)}$ are the distinct poles of y . Using the collection formula [5, Proposition 3.1] and using again Theorem 4.5, we have:

$$\begin{aligned} (M^y F)(u) &= y(u)F(u) + \int_{X_{\mathbb{R}}} f(x) \frac{d\eta(x)}{\omega(x)} K_{\zeta}(u, x) (y(x) - y(u)) \\ &= \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) f(x) y(x) \frac{d\eta(x)}{\omega(x)}. \end{aligned}$$

We note that f is a section of $L_{\zeta} \otimes \Delta$ and remains so after multiplication by a meromorphic function y . Furthermore, it is square summable with respect to the measure $d\eta$, since, by assumption, the poles of y lie outside the support of $d\eta$. \square

Corollary 4.8. *Let y be a real meromorphic function on X such that the poles of y do not belong to the support of the measure $d\eta$. Then, the multiplication model operator M^y is selfadjoint.*

Proof: The model operator M^y satisfies $\langle M^y F, G \rangle = \langle F, M^y G \rangle$ as follows from

$$\langle M^y F, G \rangle = \int_{X_{\mathbb{R}}} f(x)y(x)\overline{g(x)}\frac{d\eta(x)}{\omega(x)}$$

and from the assumption that y is a real meromorphic function. \square

Equation 4.11 immediately produces the $\mathcal{L}(\varphi)$ counterpart of [6, Theorem 4.6].

Corollary 4.9. *Let y_1 and y_2 be two meromorphic functions of degree n_1 and n_2 , respectively. Furthermore, we assume that the poles of y_1 and y_2 lie outside the support of $d\eta$ on $X_{\mathbb{R}}$. Then, M^{y_1} and M^{y_2} commute on $\mathcal{L}(\varphi)$, that is, for every $F(z) \in \mathcal{L}(\varphi)$, $M^{y_1} M^{y_2} F = M^{y_2} M^{y_1} F$ holds.*

Using the observation that R_{α}^y is just the operator $M^{\frac{1}{y(u)-\alpha}}$, we present the counterpart of (1.5), that is, an integral representation of the resolvent operator at α .

Corollary 4.10. *$\mathcal{L}(\varphi)$ is invariant under the resolvent operator R_{α}^y where α is a non-real complex number. Moreover, the resolvent operator has the integral representation:*

$$(4.12) \quad (R_{\alpha}^y F)(u) = \int_{X_{\mathbb{R}}} K_{\zeta}(u, x) \frac{f(u)}{(y(u) - \alpha)} \frac{d\eta(x)}{\omega(x)},$$

where the poles of y do not belong to the support of $d\eta$.

As another immediate corollary, we mention that any pair of resolvent operator commutes.

Corollary 4.11. *Let y_1 and y_2 be two meromorphic functions of degree n_1 and n_2 , respectively. Furthermore, assume the poles of y_1 and y_2 lie outside the support of $d\eta$ on $X_{\mathbb{R}}$ and let α and β be two elements in $\mathbb{C} \setminus \mathbb{R}$. Then the resolvent operators $R_{\alpha}^{y_1}$ and $R_{\beta}^{y_2}$ commute, i.e. for any $F(z) \in \mathcal{L}(\varphi)$ the following holds $R_{\alpha}^{y_1} R_{\beta}^{y_2} F = R_{\beta}^{y_2} R_{\alpha}^{y_1} F$.*

The counterpart of Theorem 1.2 is given below.

Theorem 4.12. *Let φ be analytic in $X \setminus X_{\mathbb{R}}$ and with positive real part. Then the following are equivalent:*

- (1) *The reproducing kernel space $\mathcal{L}(\varphi)$ is finite dimensional.*
- (2) *φ is meromorphic on X .*
- (3) *$\mathbf{L}^2(d\eta)$ is finite dimensional.*

Proof: Since $\mathcal{L}(\varphi)$ is isomorphic to $\mathbf{L}^2(d\eta)$, $\mathcal{L}(\varphi)$ is finite dimensional if and only if $\mathbf{L}^2(d\eta)$ is finite dimensional.

If $\mathbf{L}^2(d\eta)$ is finite dimensional then $d\eta$ has a finite number of atoms and hence φ is meromorphic. On the other hand assume that φ is meromorphic on X . As in the classical case, the measure in the integral representation of φ is obtained as the weak star limit of $\varphi(p) + \overline{\varphi(p)}$, and the poles of φ correspond to the atoms of $d\eta$. \square

Theorem 4.13. *Let $f, g \in \mathcal{L}(\varphi)$ and $\alpha, \beta \in \mathbb{C}$ with non-zero imaginary part. Then the following identity holds,*

$$(4.13) \quad \langle R_\alpha^y f, g \rangle - \langle f, R_\beta^y g \rangle - (\alpha - \bar{\beta}) \langle R_\alpha^y f, R_\beta^y g \rangle = 0.$$

Proof: Using (4.12), the left hand side of (4.13) can be written as

$$\begin{aligned} \langle R_\alpha^y f, g \rangle - \langle f, R_\beta^y g \rangle - (\alpha - \bar{\beta}) \langle R_\alpha^y f, R_\beta^y g \rangle &= \int_{X_{\mathbb{R}}} f(u)g(u)K_\zeta(p, u) \times \\ &\left(\frac{1}{(y(u) - \alpha)} - \frac{1}{(y(u) - \bar{\beta})} - \frac{\alpha - \bar{\beta}}{(y(u) - \alpha)(y(u) - \bar{\beta})} \right) \frac{d\eta(u)}{\omega(u)}. \end{aligned}$$

One may note that

$$\frac{1}{(y(u) - \alpha)} - \frac{1}{(y(u) - \bar{\beta})} - \frac{\alpha - \bar{\beta}}{(y(u) - \alpha)(y(u) - \bar{\beta})}$$

is identically zero, hence the result follows. \square

In fact Theorem 4.13 is an if and only if relation, and we refer the reader to [3] for the related de Branges structure theorems.

5. THE $\mathcal{L}(\varphi)$ SPACES IN THE SINGLE-VALUED CASE

Whenever an additive function φ is single-valued, the formula

$$s(p) = \frac{1 - \varphi(p)}{1 + \varphi(p)}$$

makes sense and defines a single-valued function $s(p)$. Then, the reproducing kernel associated with $s(p)$, denoted by $\mathcal{H}(s)$, is of the form

$$i(1 - s(p)s(q)^*)K_\zeta(p, \tau(q)).$$

These spaces were studied in [6] in the finite dimensional setting and in [4] in the infinite dimensional case.

We note that the multiplication operator $u \mapsto \frac{(1+\varphi(p))}{\sqrt{2}}u$ maps, as in the zero genus case, $\mathcal{H}(s)$ onto $\mathcal{L}(\varphi)$ unitarily. Hence, we may pair any

$u \in \mathcal{H}(s)$ to a function $f \in \mathbf{L}^2(d\eta)$ through the corresponding $\mathcal{L}(\varphi)$ space, such that

$$\frac{1}{\sqrt{2}}(1 + \varphi(p))u(p) = \int_{X_{\mathbb{R}}} K_{\zeta}(p, x)f(x)\frac{d\eta(x)}{\omega(x)}.$$

We denote the mapping from $\mathcal{H}(s)$ onto $\mathbf{L}^2(d\eta)$ by

$$\Lambda : u(p) \longrightarrow f(x).$$

We now turn to express the operator M^y using the operator of multiplication by y in $\mathbf{L}^2(d\eta)$.

Theorem 5.1. *Let φ be a single-valued function with positive real part on a dividing-type compact Riemann surface X and let y be a meromorphic function of degree n on X . Then, any $f \in \mathbf{L}^2(d\eta)$ satisfies*

$$(5.1) \quad (\Lambda M^y \Lambda^*)f(x) = y(x)f(x) + i \sum_{j=1}^n \frac{c_j}{1 + \varphi(p^{(j)})} K_{\zeta}(x, p^{(j)}) \left(\int_{X_{\mathbb{R}}} K_{\zeta}(p^{(j)}, p)f(p)\frac{d\eta(p)}{\omega(p)} \right),$$

where $p^{(1)}, \dots, p^{(n)}$ are the n distinct poles of y .

Proof: Let $u \in \mathcal{H}(s)$ and $f \in \mathbf{L}^2(d\eta)$ such that, $\Lambda u = f$, that is, they satisfy the relation

$$(5.2) \quad \frac{1 + \varphi(p)}{\sqrt{2}}u(p) = \int_{X_{\mathbb{R}}} K_{\zeta}(x, p)f(x)\frac{d\eta(x)}{\omega(x)}.$$

Then, multiplying both sides of (5.2) by $y(p)$, we obtain

$$(5.3) \quad \begin{aligned} y(p)\frac{1 + \varphi(p)}{\sqrt{2}}u(p) &= \int_{X_{\mathbb{R}}} y(p)K_{\zeta}(p, x)f(x)\frac{d\eta(x)}{\omega(x)} \\ &= \int_{X_{\mathbb{R}}} y(x)K_{\zeta}(p, x)f(x)\frac{d\eta(x)}{\omega(x)} + \\ &\int_{X_{\mathbb{R}}} (y(p) - y(x))K_{\zeta}(p, x)f(x)\frac{d\eta(x)}{\omega(x)}. \end{aligned}$$

Then, using the collection-type formula (see [5, Proposition 3.1, Eq. 3.5]), we have

$$(y(p) - y(q))K_{\zeta}(p, q) = - \sum_{j=1}^n \frac{c_j}{dt_j(p^{(j)})} K_{\zeta}(p, p^{(j)})K_{\zeta}(p^{(j)}, q).$$

Then (5.3) becomes:

$$(5.4) \quad \int_{X_{\mathbb{R}}} y(x) K_{\zeta}(p, x) f(x) \frac{d\eta(x)}{\omega(x)} = \frac{(1 + \varphi(p))}{\sqrt{2}} y(p) u(p) - \sum_{j=1}^n \frac{c_j K_{\zeta}(p, p^{(j)})}{dt_j(p^{(j)})} \int_{X_{\mathbb{R}}} K_{\zeta}(p^{(j)}, x) f(x) \frac{d\eta(x)}{\omega(x)}.$$

On the other hand, using the equality

$$(5.5) \quad (1 + \varphi(p)) \frac{1 + s(p)}{2} = 1.$$

Equation 5.2 becomes

$$(5.6) \quad \int_{X_{\mathbb{R}}} K_{\zeta}(x, p^{(j)}) f(x) \frac{d\eta(x)}{\omega(x)} = \frac{\sqrt{2}}{1 + s(p^{(j)})} u(p^{(j)}).$$

Now, substituting (5.5) and (5.6) in (5.4), we obtain the following calculation:

$$(5.7) \quad \begin{aligned} & \frac{\sqrt{2}}{1 + \varphi(p)} \int_{X_{\mathbb{R}}} K_{\zeta}(p, x) y(x) f(x) \frac{d\eta(x)}{\omega(x)} = \\ & = y(p) u(p) - \frac{1 + s(p)}{\sqrt{2}} \sum_{j=1}^n c_j K_{\zeta}(p, p^{(j)}) \frac{\sqrt{2} u(p^{(j)})}{1 + s(p^{(j)})} \\ & = y(p) u(p) - \sum_{j=1}^n c_j \frac{1 + s(p)}{1 + s(p^{(j)})} K_{\zeta}(p, p^{(j)}) u(p^{(j)}) \\ & = y(p) u(p) - \sum_{j=1}^n c_j K_{\zeta}(p, p^{(j)}) u(p^{(j)}) \left(1 + \frac{s(p) - s(p^{(j)})}{1 + s(p^{(j)})} \right) \\ & = (M^y u)(p) + \sum_{j=1}^n c_j \frac{s(p) - s(p^{(j)})}{\sqrt{2}} K_{\zeta}(p, p^{(j)}) u(p^{(j)}). \end{aligned}$$

On the other hand, using (5.5), we have

$$(5.8) \quad \begin{aligned} \varphi(p) - \varphi(p^{(j)}) &= \frac{2(s(p^{(j)}) - s(p))}{(1 + s(p))(1 + s(p^{(j)}))} \\ &= (1 + \varphi(p))(1 + \varphi(p^{(j)})) \frac{s(p^{(j)}) - s(p)}{2}. \end{aligned}$$

Thus, we take (5.8) and multiply it on the right by the Cauchy kernel $K_\zeta(p, p^{(j)})$ in both sides and use (4.4) to conclude

$$\begin{aligned} (1 + \varphi(p)) \frac{s(p) - s(p^{(j)})}{2} K_\zeta(p, p^{(j)}) &= \frac{\varphi(p^{(j)}) - \varphi(p)}{1 + \varphi(p^{(j)})} K_\zeta(p, p^{(j)}) \\ &= \frac{i}{1 + \varphi(p^{(j)})} \int_{X_{\mathbb{R}}} K_\zeta(p, x) K_\zeta(x, p^{(j)}) \frac{d\eta(x)}{\omega(x)}. \end{aligned}$$

Thus, (5.7) becomes

$$\begin{aligned} \frac{1 + \varphi(p)}{\sqrt{2}} (M^y u)(p) &= \int_{X_{\mathbb{R}}} K_\zeta(p, x) y(x) f(x) \frac{d\eta(x)}{\omega(x)} + i \sum_{j=1}^n \frac{c_j}{1 + \varphi(p^{(j)})} \times \\ &\quad \times \left(\int_{X_{\mathbb{R}}} K_\zeta(p, x) K_\zeta(x, p^{(j)}) \frac{d\eta(x)}{\omega(x)} \right) \left(\int_{X_{\mathbb{R}}} K_\zeta(p^{(j)}, s) f(s) \frac{d\eta(s)}{\omega(s)} \right), \end{aligned}$$

and by setting

$$\widehat{\mathbf{f}}(q) = y(q) f(q) + i \sum_{j=1}^n \frac{c_j K_\zeta(q, p^{(j)})}{1 + \varphi(p^{(j)})} \left(\int_{X_{\mathbb{R}}} K_\zeta(p^{(j)}, x) f(x) \frac{d\eta(x)}{\omega(x)} \right),$$

the identity in (5.8) becomes

$$\frac{1 + \varphi(p)}{\sqrt{2}} (M^y u)(p) = \int_{X_{\mathbb{R}}} K_\zeta(p, x) \widehat{\mathbf{f}}(x) \frac{d\eta(x)}{\omega(x)}.$$

□

Corollary 5.2. *Let φ be a single-valued function with positive real part on a dividing-type compact Riemann surface X . Furthermore, let us assume that $y(p)$ is a real meromorphic function of degree n such that $s(p^{(j)}) = 1$ for all $1 \leq j \leq n$. Then the following identity holds:*

$$(\Lambda(\Re M^y) \Lambda^*) f(p) = y(p) f(p).$$

We note that, in fact, one may assume that $s(p^{(j)})$ for all $1 \leq j \leq n$ equal to a common constant of modulus one.

Furthermore, for an arbitrary $f \in \mathbf{L}^2(d\eta)$ we set

$$\Phi_y(f) \stackrel{\text{def}}{=} \text{Col}_{1 \leq j \leq n} \left(\frac{1}{1 + \varphi(p^{(j)})} \int_{X_{\mathbb{R}}} K_\zeta(p^{(j)}, x) f(x) \frac{d\eta(x)}{\omega(x)} \right),$$

and then, for an element $d = (d_1, \dots, d_n)^t \in \mathbb{C}^n$, the adjoint operator $\Phi_y^* : \mathbb{C}^n \rightarrow \mathcal{L}(\varphi)$ is given explicitly by

$$\Phi_y^* d = \sum_{j=1}^n \frac{1}{1 + \overline{\varphi(p^{(j)})}} d_j K_\zeta(p, \tau(p^{(j)})).$$

Then, if we further use the notation

$$\sigma_y \stackrel{\text{def}}{=} \text{diag } c_j (1 + \overline{\varphi(p^{(j)})}),$$

the formula in (5.1) may be rewritten and simplified as follows

$$(\Lambda M^y \Lambda^*)f(p) = y(p)f(p) + \frac{i}{2} \Phi_y^* \sigma_y \Phi_y f,$$

while the real part of the operator M^y has the form

$$(\Lambda(\Re M^y)\Lambda^*)f(p) = y(p)f(p) + \Phi_y^* \text{diag } (\text{Im } \varphi(p^{(j)})) \Phi_y f.$$

6. SUMMARY

The table below summarizes the comparison between the $\mathcal{L}(\varphi)$ spaces in the Riemann sphere case and in a compact real Riemann surfaces setting.

	The $g = 0$ setting	The $g > 0$ setting
	$z - w$	$E(p, q)$
The Cauchy kernel	$\frac{1}{-i(z-\bar{w})}$	$K_\zeta(p, \tau(q)) \stackrel{\text{def}}{=} \frac{\vartheta[\zeta](p-\tau(q))}{i\vartheta[\zeta](0)E(p, \tau(q))}$
The Hardy space H^2	The reproducing kernel Hilbert space with kernel $\frac{1}{-i(z-\bar{w})}$	The reproducing kernel Hilbert space the with kernel $\frac{1}{-i}K_\zeta(p, \tau(q))$ where $\zeta \in T_0$.
The kernel of $\mathcal{L}(\varphi)$	$\frac{\varphi(z)+\varphi(w)^*}{-i(z-\bar{w})} = \left\langle \frac{1}{t-z}, \frac{1}{t-w} \right\rangle_{\mathbf{L}^2(d\eta)}$,	$(\varphi(p) + \varphi(q)^*) \frac{\vartheta[\zeta](p-\tau(q))}{\vartheta[\zeta](0)E(\tau(q), p)}$
Reproducing kernel in $\mathbf{L}^2(d\mu)$	$\int_{\mathbb{R}} \frac{d\eta(t)}{(t-z)(t-\bar{w})}$	$\langle K_\zeta(\tau(q), x), K_\zeta(\tau(p), x) \rangle_{\mathbf{L}^2(\frac{d\eta}{\omega})}$
The elements of $\mathcal{L}(\varphi)$	$F(z) = \int_{\mathbb{R}} \frac{f(t)d\eta(t)}{t-z}$	$\langle f, K_\zeta(\tau(p), x) \rangle = \int_{X_{\mathbb{R}}} f(x)K_\zeta(x, p)f(x) \frac{d\eta(x)}{\omega(x)}$
The Herglotz integral representation formula	$\varphi(z) = iA - iBz + i \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\eta(t)$	$\varphi(z) = \frac{\pi}{2} \int_{X_{\mathbb{R}}} \frac{[\omega_1(x) \cdots \omega_g(x)]}{\omega(x)} n(\tilde{x}) d\eta(x) + \pi i \int_{X_{\mathbb{R}}} \frac{[\omega_1(x) \cdots \omega_g(x)]}{\omega(x)} (Yp) d\eta(x) - \frac{i}{2} \int_{X_{\mathbb{R}}} \frac{\frac{\partial}{\partial x} \ln E(p, \tilde{x})}{\omega(x)} d\eta(x) + iM$
Integral representation of the model operator M^y	$(MF)(z) = \int_{\mathbb{R}} \frac{tf(t)d\eta(t)}{t-z}$	$(M^yF)(z) = \int_{X_{\mathbb{R}}} K_\zeta(u, x)f(x)y(x) \frac{d\eta(x)}{\omega(x)}$
Integral representation of the resolvent operator R_α^y	$(R_\alpha F)(z) = \int_{\mathbb{R}} \frac{f(t)d\eta(t)}{(t-z)(t-\alpha)}$	$(R_\alpha F)(p) = \int_{X_{\mathbb{R}}} K_\zeta(p, u) \frac{f(u)}{y(u) - \alpha} \frac{d\mu(u)}{\omega(u)}$

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