

Sybil-Resilient Social Choice with Partial Participation

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Abstract

Voting rules may fail to implement the will of the society when only some voters actively participate, and/or in the presence of sybil (fake or duplicate) voters. Here we aim to address social choice in the presence of sybils and voter abstention. To do so we assume the status-quo (Reality) as an ever-present distinguished alternative, and study *Reality Enforcing voting rules*, which add virtual votes in support of the status-quo. We measure the tradeoff between *safety* and *liveness* (the ability of active honest voters to maintain/change the status-quo, respectively) in a variety of domains, and show that the Reality Enforcing voting rule is optimal in this respect.

1 Introduction

Voting procedures are a simple and widely used way to aggregate the preferences of multiple individuals. Voting, however, can truly reflect the will of the society only insofar as all eligible people in the society—and only them—vote.

The problem of partial participation in online voting is particularly acute, as online voting often exhibit very low participation rates [Christian Schaupp and Carter, 2005; Jönsson and Örnebring, 2011]. For example, in the 2006 Cambridge MA participatory budgeting program, only 7.5% out of ~64000 eligible voters actually participated [Mundt, 2017].

The orthogonal problem of sybil votes has received much attention (see Related Work). Recently, Shahaf et al. [2019] studied democratic decision-making in the presence of sybil voters. The key concept in their approach to sybil resilience is the use of the *status-quo*, or *Reality*, as the anchor of sybil resilience, following [Shapiro and Talmon, 2018]. Shahaf et al. [2019] formalize the properties of *sybil safety*—the inability of sybils to change the status-quo against the will of the honest voters, and *sybil liveness*—the ability of the honest voters to change the status-quo despite the existence of sybils. Then, they design specialized sybil-resilient voting rules for the following

social choice settings: (1) voting on two alternatives; (2) multiple alternatives; and (3) an interval. A major limitation of their work is the assumption of full participation by the genuine (non-sybil) voters.

Our main aim in this paper is to develop mechanisms that mitigate the adverse effects of partial participation to sybil resilience. To this end, we adopt and extend the safety-liveness framework suggested in [Shahaf *et al.*, 2019] to deal with partial participation in general domains. Thus, we aim at finding voting rules that satisfy sybil-safety but are not too conservative, even when some genuine voters do not vote.

1.1 Related Work

Sybil attacks have been amply studied in the literature of computational social choice (see e.g. [Tran *et al.*, 2009; Conitzer and Yokoo, 2010; Conitzer *et al.*, 2010]), mainly showing impossibility results (and sometimes positive results) on the design of *false-name proof* voting rules, i.e., rules where clones cannot affect the outcome. These results are not applicable in our safety/liveness model. A related challenge is keeping the fraction of sybils in online communities low, which may be possible via identification and eradication techniques (see, e.g., [Alvisi *et al.*, 2013]).

There is extensive work in the social choice literature on the strategic justification of partial participation/abstention, going back to the “paradox of nonvoting” [Riker and Ordeshook, 1968; Owen and Grofman, 1984; Desmedt and Elkind, 2010]. Voting with a *random set* of active voters has been widely considered, and boils down to problems of statistical estimation. See e.g. [Regenwetter *et al.*, 2006; Dey and Bhattacharyya, 2015]. Other works consider ways to elicit the preferences of specific voters in order to reduce communication complexity [Conitzer and Sandholm, 2005]. Yet we are unaware of works that consider resilience to arbitrary partial participation.

Some of our ideas and analysis are based on (nontransitive) vote delegation [Brill and Talmon, 2018; Cohensius *et al.*, 2017; Green-Armytage, 2015; Kahng *et al.*, 2018].

1.2 Summary of Our Results

On the conceptual side, we generalize the definitions of safety and liveness of Shahaf *et al.* [2019] to account for both sybils and partial participation, as well as to general domains. We show that the results of Shahaf *et al.* [2019] in two previously studied domains (single proposal and interval) can be unified under the domain-independent Reality-Enforcing (*RE*) mechanism, a mechanism that simply adds fictitious votes in favor of the status-quo. In particular, we provide a simple reduction between the two domains.

On the technical side, our contributions are as follows: We provide a tight analysis of the RE mechanism for both domains to uncover the exact fraction of sybils and abstentions that the mechanism can accommodate. These results completely generalize the ones in [Shahaf *et al.*, 2019]. We complement our analysis with a lower bound, showing that no mechanism can do better under arbitrary partial participation. We extend the results to multiple alternatives, multiple referenda, and to a random participation model (where an improved tradeoff is obtained). Finally, we consider a “proxy voting” situation where few active genuine voters are selected at random, with other genuine

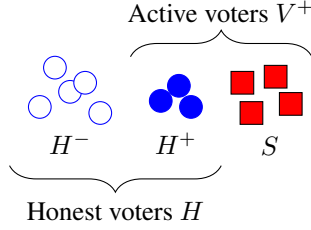


Figure 1: The general voting population.

voters delegating their vote to the nearest active voter. By leveraging a recent result of Cohensius et al. [2017], we show that the RE mechanism combined with proxy voting has an excellent safety-liveness tradeoff, even with only a small fraction of active genuine voters.

2 Preliminaries

We consider voting situations with a set A of alternatives, one of which is the current *reality*, or *status-quo*, denoted by r . There is a set V of n voters. A voting rule is a function taking the n votes and returning an outcome in A . The social choice settings we consider in this paper are such that each voter votes by picking an alternative and the aggregated outcome is also an alternative; thus, a voting rule \mathcal{R} is a sequence of functions $\mathcal{R}^n : A^n \rightarrow A$, for all $n \in \mathbb{N}$.

2.1 Formal Model of Partial Participation and Sybils

Previous work considered either situations of partial participation [Cohensius *et al.*, 2017] but without sybils or situations with sybils but assuming full participation [Shahaf *et al.*, 2019]. Here we extend and generalize these models by considering both sybil and partial participation.

The set of voters V is partitioned into a set of *honest* (i.e., genuine; non-sybil) voters H and a set of *sybil* voters S ; so, $V = H \cup S$ with $H \cap S = \emptyset$. Ideally, we would like our voting rules to reflect only the preferences of the honest voters, but without them knowing who is honest and who is sybil, and when not all honest voters vote.

Active and passive voters The set of honest voters, H , is further partitioned into $H = H^+ \cup H^-$ (with $H^+ \cap H^- = \emptyset$), where H^+ is the non-empty set of honest voters who did cast a vote, and are thus labeled by their vote, and H^- is the set of honest voters who did not cast a vote. We refer to the voters in H^+ as *active honest voters* and to the voters in H^- as *passive honest voters*, or *passive voters* in short.

As this will correspond to the worst-case, we will assume, w.l.o.g, that all sybils vote. Thus, both the active honest voters and the sybils are active. Denote $V^+ := H^+ \cup S$. See Figure 1.

Further notation In many places it will be convenient to refer to the *fraction* of some set of voters rather than to their absolute size. For any subset of voters $U \subseteq V$, we denote by the small letter $u := \frac{|U|}{|V|}$ the relative size of this set to the entire population.

We denote by $U_a \subseteq U$ the subset of U voters who vote for alternative $a \in A$, and by $u_a = \frac{|U_a|}{|V|}$ their relative fraction.

Lastly, we use a special notation for the fraction of sybils σ and the fraction of inactive voters μ , as these parameters are used frequently throughout the paper. Formally, $\sigma := s = \frac{|S|}{|V|}$; and $\mu := h^- = \frac{|H^-|}{|V|}$.

2.2 Safety and Liveness

Suppose we have some preferred voting rule \mathcal{G} , for the “standard” setting without sybils and with full participation. This may be due to favorable axiomatic or social properties of \mathcal{G} , because of its simplicity, due to legacy, or for any other reason. Ideally, we would like to always get outcome $\mathcal{G}(H)$, that is, the result of all honest voters voting under \mathcal{G} . However, if we use \mathcal{G} in a straight-forward way, then the outcome may be distorted due to the existence of sybil votes, due to the partial participation, or both.

Example 1 (running example). As a simple example, say that we would like to use Plurality for the voting population $\{v_1, v_2, v_3, v_4, v_5\}$, in which v_1 and v_2 are sybils voting for candidate a ; v_3, v_4 , and v_5 are honest supporters of r where v_3 and v_4 abstain. Then, Plurality on the active voters $\{v_1, v_2, v_5\}$ would select p even though a majority of the honest voters wish r to be selected.

Intuitively, safety means that the outcome shall be somewhere between the desired outcome and some current status quo outcome (so, informally speaking, the sybils are not able to push the society to dangerous places, even when not all honest voters participate); and liveness means that a sufficient fraction of active honest voters have the power to change the status-quo (so, informally speaking, the voting rule still give some power to the honest voters to decide when to move from the status quo).

To describe our formal definitions, we need some notation. First, our formal definitions of safety and liveness apply to arbitrary domains, any aggregation rule, and various scenarios (e.g., partial participation, sybils). Following Reality-Aware Social Choice [Shapiro and Talmon, 2018], we assume that a distinguished alternative, the Reality, r , always exists. We use r for natural tie-breaking, where all ties are settled towards the alternative closer to r . We will use variants of our running example above to demonstrate the definitions as we go along.

Outcome range We first provide the following notation, in which the parameter γ is used as a notion of approximation. In our definitions, we consider some profile $V = H \cup S = H^+ \cup H^- \cup S$ of voters voting on a set of alternatives A with $r \in A$.

Definition 1 (Outcome Range). Let \mathcal{R} be an aggregation rule. Then, for a parameter $\gamma \geq 0$, we define:

$$\overline{\mathcal{R}}_\gamma(V) := \{\mathcal{R}(H' \cup S) : \exists H', |H'| \geq |H|, |H' \setminus H| \leq \gamma|H|\}.$$

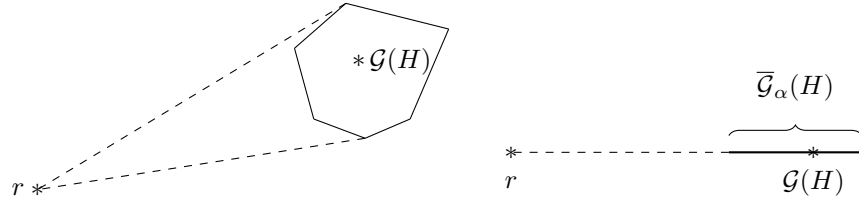


Figure 2: A demonstration of the α -safety property in some 2-dimensional metric space (left), and on a 1-dimensional space (right). The area inside the solid line is $\bar{\mathcal{G}}_\alpha(H)$. The area inside solid or dashed lines is $B(r; \bar{\mathcal{G}}_\alpha(H))$.

For $\gamma \in [0, 1]$, this means that $\bar{\mathcal{R}}_\gamma(V)$ contains all outcomes that can be obtained by replacing a fraction of at most γ honest voters. Higher values $\gamma > 1$ mean that we may replace all honest voters and, furthermore, add additional $(1 - \gamma)|H|$ voters. Note that $\bar{\mathcal{R}}_\gamma$ is superset-monotone in γ (i.e., $\bar{\mathcal{R}}_\gamma(V) \subseteq \bar{\mathcal{R}}_{\gamma'}(V)$ for $\gamma \leq \gamma'$), with $\bar{\mathcal{R}}_0(V)$ being the singleton $\mathcal{R}(V)$.

Betweenness As we wish to deal with many social choice settings and have our definitions general, we consider the notion of *betweenness*. Indeed, any metric space (A, δ) induces a natural trinary relation of *betweenness* of distinct points, where b is *between* a and c if $\delta(a, b) + \delta(b, c) = \delta(a, c)$ (first defined by Menger [1928]; see [Chvátal, 2004]).

Definition 2 (Between set). For $x, y \in A$, $B(x, y) \subseteq A$ is the set of all points that are *between* x and y , including $\{x, y\}$. We define $B(x; Y) := \bigcup_{y \in Y} B(x, y)$.

In particular:

- in a discrete unordered set A , we have $B(x, y) = \{x, y\}$;
- in multiple referenda (with d binary issues and the Hamming distance) $A = \{0, 1\}^d$, $B(x, y)$ is the smallest *box* containing both of x, y [Nehring and Puppe, 2007];
- on the real line, $B(x, y)$ is the smallest *interval* containing both x, y .

Safety We are ready to define our notion of safety. In particular, we measure safety w.r.t some given base rule \mathcal{G} , which represents the desired outcome given no sybils and full participation. While in some cases there may be an “obvious” base rule (e.g. majority in the binary domain), this is not always the case and thus we require the base rule to be specified explicitly.

Definition 3 (Safety). An aggregation rule \mathcal{R} is α -safe w.r.t the base rule \mathcal{G} and population $V = H \cup S$, if it holds that $\mathcal{R}(V) \in B(r; \bar{\mathcal{G}}_\alpha(H))$.

Importantly, for $\alpha = 0$, we have that an aggregation rule is 0-safe if it always hits between r and the correct outcome $\mathcal{G}(H)$. For general $\alpha > 0$, indeed α -safety relaxes that notion by using both betweenness and the outcome range as a notion of

approximation (see Fig. 2). E.g., on the real line, $B(r; \overline{\mathcal{G}}_\alpha(H))$ is the interval between the status-quo and the farthest point that is still in the outcome range. Note that returning the status-quo r is always safe. The base rule is needed for the definition of safety in order to have a notion of a “correct” outcome (put differently, the base rule applied to the honest votes defines what is the desired goal of the aggregation).

Liveness, however, does not depend on a base rule.

Definition 4 (Liveness). An aggregation rule \mathcal{R} is β -live w.r.t. population V , if for any $a \in A \setminus \{r\}$, it holds that $a \in \overline{\mathcal{R}}_\beta(V)$.

We say that a rule \mathcal{R} is safe [resp., live], if it is safe [live] w.r.t. any population V .

So, a rule is live if any outcome can be reached by modifying not-too-many (in particular, β -fraction of) honest voters. In particular: 1-liveness means that the rule is *onto* (sometimes called *citizen sovereignty* [Moulin, 2014]), i.e., the honest voters can enforce any outcome. In other words, sybils have no veto power, which is the main liveness requirement we consider. Lower values of β can also be of interest:

- $\beta < 1$ means that the rule satisfies the *no veto power* requirement [Maskin, 1999] w.r.t. the honest voters.
- $\beta \leq 0.5$ means that the rule satisfies the weak majority requirement.
- $\overline{\mathcal{R}}_0(V)$ is the singleton $\mathcal{R}(V)$, thus 0-liveness is satisfied only in the degenerate case of $A = \{r\}$.

Remark 1. Our definitions generalize those of Shahaf et al. [2019]; when $\mu = 0$ (that is, with full participation), their definitions of safety and liveness coincide with our 0-safety and 1-liveness, respectively.

The following example considers the majority rule MJ .

Example 2 (Majority under full participation). Let $V = \{x_1, x_2, x_3, x_4, x_5\}$, where $H = \{x_1, x_2, x_3\}$ are honest voters, $S = \{x_4, x_5\}$ are sybils, and $A = \{p, r\}$. Thus, $\sigma = \frac{2}{5}$ (as there are 2 sybils out of 5 voters). Consider the base rule $\mathcal{G} = MJ$ which returns p if it has a simple majority, else r . Next, we show that for any $\alpha < \frac{1}{3}$, MJ is not α -safe w.r.t itself even under full participation. To this end, suppose that x_1 and x_2 support r whereas x_3 supports p . As $\alpha|H| < 1$, we cannot change to vote of any voter in $\overline{MJ}_\alpha(H)$. Now, observe that $\overline{MJ}_\alpha(H) = \{r\}$. Then, α -safety requires that $MJ(V) \in B(r; \overline{MJ}_\alpha(H)) = \{r\}$, however, if the sybils both vote for p , then $MJ(V) = p \notin \{r\}$.

Next we show that, at least for the case of $|H| = 3$ and $|S| = 2$ as above, we have that MJ is α -safe w.r.t itself for each $\alpha \geq \frac{1}{3}$. To this end, note that if all honest voters vote for r , then $MJ(V) = r$, and returning r is always safe; otherwise, if there is at least one honest vote for p , then, by setting $H' := \{p, p, r\}$ (possible for $\alpha \geq \frac{1}{3}$, see Fig. 3(b)), we get that $p \in \overline{MJ}_\alpha(H)$, thus we can return p .

What about liveness? For any 5-voter profile V as above, we can define an alternative profile H' where all 3 = $1 \cdot |H|$ honest voters vote for p , and $MJ(H' \cup S) = p$. Thus $p \in \overline{MJ}_1(V)$, and MJ is 1-live. Also, any lower value $\alpha < 1$ means we can change the votes of at most two honest voters in H' . By our tie-breaking rule this means $MJ(H' \cup S) = r$ and thus MJ is not β -live for any $\beta < 1$.

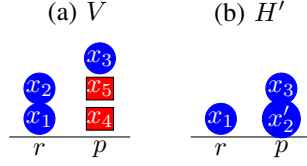


Figure 3: Red squares are sybils, blue circles are active honest voters. (a) The population V used in Examples 2 and 3. (b) A set $|H'|$ that demonstrates the $\frac{1}{3}$ -safety property in Example 2.

For the next example, we first consider another rule that takes the status-quo into account [Shahaf *et al.*, 2019]:

Definition 5. Suppose $A = \{r, p\}$. Then, the τ -Supermajority rule (τ -SMJ) selects p if $v_p > \frac{1}{2} + \tau$ voters vote for p (recall that v_p is the fraction of voters voting for p); otherwise, it selects r .

Example 3 (Supermajority under full participation). Consider the same profile of votes from Example 2, but with the supermajority rule $\mathcal{R} = 0.4$ -SMJ. This rule returns p only if more than 90% of votes are for p , and otherwise returns r . Note that for profiles with five voters this requires a unanimous vote for p . For this particular V , we do not observe violation of 0-safety for $\mathcal{R} = 0.4$ -SMJ w.r.t $\mathcal{G} = MJ$; to see why, note that, if $\mathcal{R}(V) = p$, then all 3 honest voters must vote p , so $p \in \overline{MJ}_0(H) \subseteq B(r; \overline{MJ}_0(H))$, while if $\mathcal{R}(V) = r$ then safety cannot be violated. We can also see that under the rule 0.4-SMJ, letting the two sybils vote r shows a violation of 1-liveness.

Rule 0.4-SMJ is $6\frac{1}{3}$ -live, since we can construct a set H' of $19 = (6\frac{1}{3})|H|$ voters for p . It cannot be β -live for $\beta < 6\frac{1}{3}$, since fewer than 19 voters on p will not suffice for a supermajority if the two sybils vote r .

2.3 Safety and Liveness under Partial Participation

Next, we introduce a formal definition that captures the effect of passive voters. Recall that $V = H^+ \cup H^- \cup S$ and $V^+ = H^+ \cup S$. The following definition captures the idea that the voting rule can only see the active votes and cannot distinguish between genuine votes and sybil votes.

Definition 6. $\mathcal{R}^+(V)$ returns $\mathcal{R}(V^+)$ for any input V .

Suppose we run a voting rule \mathcal{R} on population V (say with 100 voters), but 30 of them remain passive. Then in practice we compute $\mathcal{R}(V^+)$, i.e., taking into account only the 70 active voters, and this is equivalent to running rule \mathcal{R}^+ on the entire population. Also, we assume naturally that no rule can distinguish between voters in S and in H^+ .

Therefore, when considering a voting rule \mathcal{R} under partial participation, we care about the safety and liveness properties of \mathcal{R}^+ . Formally:

- \mathcal{R}^+ is α -safe w.r.t. \mathcal{G} if $\mathcal{R}(V^+) \in B(r; \overline{\mathcal{G}}_\alpha(H))$ for all V .

- \mathcal{R}^+ is β -live if $a \in \overline{\mathcal{R}}_\beta(V^+)$ for all V and $a \in A$.¹

Example 4 (Majority under Partial Participation). We extend Example 2 by partitioning the set of honest voters into one active voter $H^+ = \{x_1\}$ and two passive voters $H^- = \{x_2, x_3\}$. As in Example 2, x_4 and x_5 are sybils, so $\sigma = \frac{2}{5}, \mu = \frac{2}{5}$ (see Fig. 4(a)). The modified rule MJ^+ considers only the active voters. While Example 2 shows that MJ is $\frac{1}{3}$ -safe w.r.t MJ , this is not true under partial participation, as it is possible that all three honest voters prefer r but $MJ^+(V) = MJ(\{x_1\} \cup S) = p$. In this case we can only show that MJ^+ is α -safe w.r.t MJ for $\alpha \geq \frac{2}{3}$ (as we need to change two honest voters to include p in the range $\overline{MJ}_\alpha(H)$, see Fig. 4(b)).

The MJ^+ rule is 3-live. To see that $\beta \leq 3$, observe that we can define $H' = 3 = 3|H^+|$ voters for p , and then $MJ(H' \cup S) = p$ and $MJ(H' \cup S) \in \overline{MJ}_3(V^+)$.

For the lower bound, note that if $\{x_1, x_4, x_5\}$ all vote for r , then for any $\beta < 3$ we can only place x_1 and at most one additional vote x' on p . Thus each alternative has two votes, and by our tie-breaking rule $\overline{MJ}_\beta(V^+) = \{r\}$ only.

Example 5 (Supermajority under partial participation). Consider the same population as in Example 4, but with the rule $\mathcal{R}^+ = 0.4\text{-}SMJ^+$, which requires a a supermajority of 0.4 (that is, over 90% of all votes) to select p .

SMJ^+ is $\frac{1}{3}$ -safe since $0.4\text{-}SMJ^+(V) = 0.4\text{-}SMJ(\{x_1\} \cup S) = p$ means that x_1 must vote p and we only need to change one of the three honest voters.

Similarly, $0.4\text{-}SMJ^+$ is 19-live since we need $19 = 19|H^+|$ votes for p , so that the two sybils would account for strictly less than 10% of votes.

Table 1 summarizes the safety and liveness properties of the Majority rule and of 90%-supermajority for the particular populations we considered in our examples. We can see, and it is indeed intuitively appealing, that requiring a supermajority boosts safety, but hurts liveness. We also see that both safety and liveness may be substantially hit (i.e., their values increase) due to partial participation.

2.4 The General Reality-Enforcing Mechanism

For a base voting rule \mathcal{R} , we define a parameterized version that adds votes to the status-quo. As we shall see, this parameterized rule achieves optimal safety and liveness tradeoff. A formal definition follows.

Definition 7 (Reality-enforcing mechanism). Let \mathcal{R} be a rule. Then, we define $\tau\text{-RE-}\mathcal{R}(V) := \mathcal{R}(V \cup Q)$, where Q is a set of $\tau|V|$ voters voting for r .

We are interested in the safety and liveness properties of $\tau\text{-RE-}\mathcal{R}^+$. Intuitively, a higher value of τ increases safety as it makes it harder to push society from the status-quo; but it may hurt liveness. In Example 4, the $\frac{2}{3}\text{-RE-}MJ^+$ mechanism will add $\frac{2}{3}|V^+| = 2$ ‘virtual votes’ to r (see Fig. 4(c)). Thus $\frac{2}{3}\text{-RE-}MJ^+(V) = MJ^+(V \cup Q) = MJ(H^+ \cup S \cup Q) = MJ(\{r, p, p, r, r\}) = r$. Note that this entails 0-safety.

¹Note that $\overline{\mathcal{R}}_\beta^+(V)$ does not equal $\overline{\mathcal{R}}_\beta(V^+)$. In particular the former is not well-defined for $\beta > 1$, since we would have to specify which of the honest voters we add to H^+ is active. Our definition avoids this complication altogether.

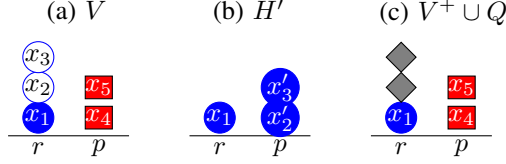


Figure 4: Red squares are sybils, full/hollow circles are active/passive honest voters, respectively. (a) The population used in Examples 4 and 5. Note that $MJ(V^+) = p$ whereas $MJ(H) = r$. (b) A set H' that demonstrates the $\frac{2}{3}$ -safety property. (c) The rule $\frac{2}{3}$ -RE-MJ adds $\frac{2}{3}|V^+| = 2$ virtual voters on r (gray diamonds).

participation	Population					Safety (α)		Liveness (β)	
	$ H^+ $	$ H^- $	$ S $	μ	σ	MJ	0.4-SMJ	MJ	0.4-SMJ
Full	3	0	2	0	$\frac{2}{5}$	$\frac{1}{3}$	0	1	$6\frac{1}{3}$
Partial	1	2	2	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{3}$	$\frac{1}{3}$	3	19

Table 1: A summary of Examples 1-4. All safety properties are w.r.t. the base rule $\mathcal{G} = MJ$, and also w.r.t. the particular profile V described in the examples.

Throughout the paper we particularly show that the Reality-enforcing mechanism generalizes both the supermajority mechanism (in the single proposal domain) and the suppress-outer-votes mechanism (in the interval domain) of Shahaf et al. [2019].

3 Single Proposal against the Status Quo

The first social choice setting we consider is a binary vote: voting on a single proposal p against the status-quo r , as in Examples 2-5. Here, it is natural to use Majority as a base rule; in fact, May's theorem [May, 1952] shows that it is the *only* natural voting rule to consider.²

Denote the proposal by p and the status-quo by r , so that $A = \{p, r\}$. Using the Majority rule MJ as the base rule \mathcal{G} , we have that $\mathcal{G}(H) = p$ if there are strictly more honest voters for p and r otherwise (recall that, following Reality-aware Social Choice [Shapiro and Talmon, 2018], we break ties in favor of the status quo).

Full participation For the case when sybils are present and all voters participate, Shahaf et al. [2019] studied the Reality-aware τ -Supermajority rule (see Def. 5).

Theorem 1 (Shahaf et al. [2019, Theorem 1 and Corollary 1]). *If $\mu = 0$ (full participation) and $\sigma \leq 2\tau$, then τ -SMJ is both 0-safe and 1-live w.r.t. Majority as the base rule.*

By applying the RE mechanism to the Majority base rule \mathcal{R} , we get the τ -RE-MJ rule, which returns the majority opinion after adding $\tau|V|$ votes for the status-quo r .

²Another, informal justification for the Majority rule comes from the Condorcet Jury Theorem, although in this paper we do not assume the existence of a ground truth.

Observation 2. For $A = \{r, p\}$ and any $\tau \geq 0$, the 2τ -RE-MJ rule and the τ -SMJ rule of Shahaf et al. [2019] coincide.

Proof. Consider the fraction v_p of votes for p . The claim follows since $v_p > v_r + q_r = (1 - v_p) + 2\tau$ (i.e. τ -RE-MJ selects p) iff $v_p > \frac{1}{2} + \tau$ (i.e. p has supermajority). \square

3.1 Arbitrary Participation

We capture the full tradeoff between safety and liveness: In particular, we analyse, for given values of σ , τ , and μ , what values of α and β allow us to obtain α -safety and β -liveness. Intuitively, higher values of α and β are easier to obtain, as in all minimization problems (as α and β are approximation notions).

Theorems 3 and 4 are our main results in the paper, and most other results are either derived from them, show their tightness, or extend them to variations of the model.

Theorem 3 (Safety). *The τ -RE-MJ⁺ voting rule is α -safe w.r.t Majority as the base rule if and only if $\alpha \geq \frac{1+\sigma-(1+\tau)(1-\mu)}{2(1-\sigma)}$.*

Proof. Consider a given profile V . If τ -RE-MJ⁺(V) = r or $p \in \overline{MJ}_\alpha(H)$ then there is no violation of α -safety and we are done. Thus, assume that τ -RE-MJ⁺(V) = $p \neq r$. Recall that h_p^+ denotes the fraction of active honest voters voting for p . W.l.o.g. we may assume that all of S vote for p , since if profile V violates α -safety, we can define a new profile V' , by switching all S agents who vote for r with p voters, and we would still have τ -RE-MJ⁺(V) = p (and $\overline{MJ}_\alpha(H)$ is unaffected) and thus there is still a violation in V' (so, intuitively, profiles in which all sybils vote for p are the hardest case for keeping safety). Similarly, we assume w.l.o.g that all of H^- vote for r , thus $h_r = h^- + h_r^+$, $h_p = h_p^+$ (again, profiles in which all passive voters vote for r are the hardest case for keeping safety, as safety is defined w.r.t all honest voters); so, the fraction of active honest voters voting for r is $h_r^+ = 1 - \sigma - \mu - h_p^+$. Since τ -RE-MJ⁺(V) = p , we have that

$$\begin{aligned} h_p^+ + \sigma &= v_p^+ > v_r^+ + q = h_r^+ + q = h^+ - h_p^+ + q \\ &= (1 - \mu - \sigma - h_p^+) + \tau(1 - \mu), \text{ and thus} \\ 2h_p^+ &> (1 + \tau)(1 - \mu) - 2\sigma. \end{aligned} \tag{1}$$

To show that $p \in \overline{MJ}_\alpha(H)$, which would show α -safety, it is left to show that we can change the votes of $\alpha \cdot h$ honest voters from r to p , to create a new profile H' where p has a strict majority of honest votes. Denote

$$\alpha' = \alpha h = \alpha(1 - \sigma) \geq \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2}. \tag{2}$$

Indeed, after moving α' votes, r has

$$h'_r = h_r - \alpha' = h - h_p - \alpha' = 1 - \sigma - h_p^+ - \alpha'$$

honest votes, whereas p has $h'_p = h_p^+ + \alpha'$ honest votes. Therefore, we have that

$$\begin{aligned}
h'_p - h'_r &= (h_p^+ + \alpha') - (1 - \sigma - h_p^+ - \alpha') \\
&= 2(h_p^+ + \alpha') - (1 - \sigma) \\
&\geq 2h_p^+ + (1 + \sigma - (1 + \tau)(1 - \mu)) - (1 - \sigma) && \text{(By Eq. (2))} \\
&> (1 - \sigma) - (1 - \sigma) = 0. && \text{(By Eq. (1))}
\end{aligned}$$

So, there are strictly more honest votes for p than for r .

In the other direction (i.e. to show tightness of the bound), consider τ, σ, μ and $\alpha < \frac{1+\sigma-(1+\tau)(1-\mu)}{2(1-\sigma)}$: First set $\varepsilon = \frac{1+\sigma-(1+\tau)(1-\mu)}{2(1-\sigma)} - \alpha$. Next, set $h_p^+ = \frac{(1+\tau)(1-\mu)-2\sigma}{2} + \varepsilon'$, where $\varepsilon' \in (0, \frac{\varepsilon}{1-\sigma})$. All σ sybils vote for p , and all μ inactive honest voters vote for r .

It is left to show that (a) $\overline{MJ}_\alpha(H) = \{r\}$ (i.e. r is the only safe outcome); and that (b) $\tau\text{-RE-MJ}^+(V) = p$ (details omitted). For (a), consider any honest profile H' such that $|H' \setminus H| \leq \alpha h$. In the best case, we have that $h'_p \leq h_p + \alpha h$ and $h'_r \geq h_r - \alpha h$. Indeed,

$$\begin{aligned}
h'_p - h'_r &\leq h_p - h_r + 2\alpha h = h_p - (h - h_p) + 2\alpha h \\
&= 2h_p - (1 - \sigma) + 2\alpha(1 - \sigma) \\
&= 2h_p^+ - (1 - \sigma) + 2\alpha(1 - \sigma) \\
&= [(1 + \tau)(1 - \mu) - 2\sigma + 2\varepsilon'] - (1 - \sigma) \\
&\quad + [(1 + \sigma) - (1 + \tau)(1 - \mu) + 2\varepsilon(1 - \sigma)] \\
&= 2\varepsilon' - 2\varepsilon(1 - \sigma) < 0,
\end{aligned}$$

which shows that $MJ(H') = r$ as required.

For (b), we can see that

$$\begin{aligned}
v_p^+ - (v_r^+ + q) &= (h_p^+ + \sigma) - ((h^+ - h_p^+) + \tau v^+) \\
&= 2h_p^+ - h^+ - v^+ \tau + \sigma \\
&= 2h_p^+ - (1 - \sigma - \mu) - (1 - \mu)\tau - \sigma \\
&= 2h_p^+ - (1 - \mu)(1 + \tau) - 2\sigma \\
&= 2\varepsilon' > 0, && \text{(by definition of } h_p^+)
\end{aligned}$$

which shows that $\tau\text{-RE-MJ}^+(V) = p$ and thereby completes the proof. \square

Fig. 5(a) shows an example where there is a large majority of honest voters for r , and yet $\tau\text{-RE-MJ}^+$ selects p . Thus, this profile implies a violation of α -safety whenever $\alpha < \frac{h_r - h_p}{2}$. Otherwise, we can define an profile H' where $\alpha|H|$ honest voters switch from r to p and get $MJ(H' \cup S) = p$.

Theorem 4 (Liveness). *The $\tau\text{-RE-MJ}^+$ voting rule is β -live if and only if $\beta > \frac{(1-\mu)(1+\tau)}{2(1-\sigma-\mu)}$.*

Proof. Since any vote for r reduces liveness, w.l.o.g all voters vote for r . There are $h^+ = 1 - \mu - \sigma$ active honest voters (all vote for r). Suppose we create a new profile

\bar{V} by moving a fraction of β votes from r to p , then p has $\bar{v}_p^+ = \bar{h}_p^+ = \beta(1 - \mu - \sigma)$ votes.

In contrast, r has $\bar{h}_r^+ = h^+ - \bar{h}_p^+ = 1 - \mu - \sigma - \bar{h}_p^+$ active honest votes remaining, plus σ sybils. The τ -RE-MJ⁺ mechanism adds $\tau(1 - \mu)$ votes so the total support for r is

$$\bar{v}_r^+ = (1 - \mu - \sigma - \bar{h}_p^+) + \sigma + \tau(1 - \mu) = (1 + \tau)(1 - \mu) - \bar{h}_p^+.$$

Since liveness requires $\bar{v}_p^+ > \bar{v}_r^+$, we get a tight bound of $2\bar{h}_p^+ > (1 + \tau)(1 - \mu)$, or, equivalently,

$$\beta = \frac{\bar{h}_p^+}{1 - \mu - \sigma} > \frac{(1 + \tau)(1 - \mu)}{2(1 - \mu - \sigma)},$$

as required. \square

Remark 2 (Mechanism design perspective). The analysis of the α -safety and β -liveness of τ -RE-MJ⁺ for given values of σ and μ implies a different point of view: Indeed, in practical situations, the values of α -safety and β -liveness might be decided by a user of the system (a stricter user would require smaller values); then, given some estimations of σ and μ (μ can usually be known exactly, while to estimate σ one can use, e.g., sampling techniques as described by Shahaf et al. 2019), the analysis above can be used to infer what value of τ the user shall choose for the τ -RE-MJ⁺ mechanism to achieve the desired levels of safety and liveness. Below, as important examples, we consider the special cases of 0-safety and 1-liveness.

Corollary 5. *The following hold:*

- τ -RE-MJ⁺ is 0-safe w.r.t MJ iff $\tau \geq \frac{1+\sigma}{1-\mu} - 1$.
- τ -RE-MJ⁺ is 1-live iff $\tau < \frac{2(1-\sigma-\mu)}{1-\mu} - 1$.
- We can get both iff $3\sigma + 2\mu < 1$.

E.g., with 20% sybils and 20% passives, or with 10% sybils and 35% passives, we can get both 0-safety and 1-liveness by respecting the condition of the above corollary.

We complement our analysis with the following lower bound, showing the optimality of τ -RE-MJ⁺.

Theorem 6. *There is no mechanism \mathcal{R} such that \mathcal{R}^+ is both 0-safe and 1-live when $3\sigma + 2\mu \geq 1$.³*

Proof. Assume towards a contradiction that such a mechanism \mathcal{R} exists. By 1-liveness, there is a profile V s.t. all sybils are voting for r , and $\mathcal{R}^+(V) = \mathcal{R}(S \cup H^+) = p$. The total number of active voters for p is h_p^+ . Note that $h_p^+ \leq h^+ = 1 - \mu - \sigma$.

Now, consider a profile $\bar{V} = \bar{S} \cup \bar{H}^+ \cup \bar{H}^-$, where $|\bar{S}| = |S|$, $|\bar{H}^+| = |H^+|$, $|\bar{H}^-| = |H^-|$, so σ and μ are the same as in V . Set $\bar{s}_p := \min\{h_p^+, \sigma\}$ sybils to vote for p , as well as exactly $\bar{h}_p^+ := h_p^+ - \bar{s}_p$ honest voters. All other voters vote for r (including

³ We assume that there is at least one honest voter, otherwise safety is meaningless.

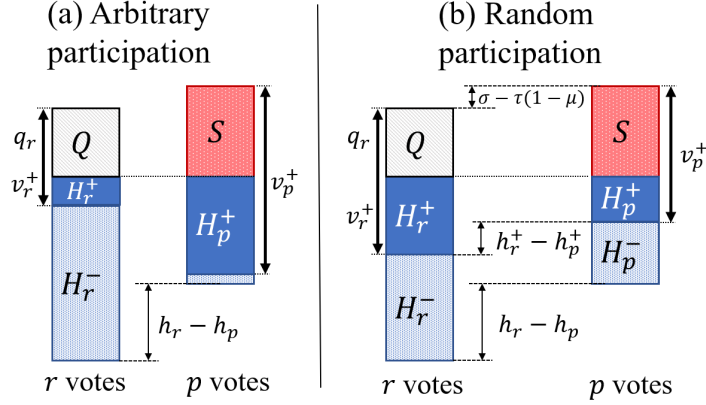


Figure 5: An example of voting profiles with the same σ, μ parameters under arbitrary partial participation (a), and under random partial participation (b). The thick arrows show the total amount of active votes for each alternative.

all inactive honest voters). Since $\bar{v}_p^+ = v_p^+$ and $\bar{v}_r^+ = v_r^+$, the profiles V and \bar{V} are indistinguishable for \mathcal{R}^+ , and we have $\mathcal{R}^+(\bar{V}) = p$ as well.

We will show that $\bar{h}_r \geq \bar{h}_p$, which entails a violation of 0-safety. Suppose first that $\sigma < h_p^+$. Then

$$\begin{aligned}
\bar{h}_r - \bar{h}_p &= (\bar{h}^- + \bar{h}_r^+) - \bar{h}_p^+ = (\bar{h}^- + \bar{h}^+ - \bar{h}_p^+) - \bar{h}_p \\
&= \bar{h}^- + \bar{h}^+ - 2\bar{h}_p^+ = \mu + (1 - \sigma - \mu) - 2\bar{h}_p^+ \\
&= 1 - \sigma - 2(h_p^+ - \bar{s}_p) = 1 - \sigma - 2(h_p^+ - \sigma) \\
&= 1 + \sigma - 2h_p^+ \geq 1 + \sigma - 2(1 - \mu - \sigma) \\
&= 3\sigma + 2\mu - 1 \geq 0,
\end{aligned}$$

where the last inequality is by the premise of the theorem.

If $\sigma \geq h_p^+$, then \bar{V} contains no honest voters for p at all, which means $\bar{h}_r > \bar{h}_p$. \square

3.2 Random Partial Participation—Nonatomic Population

The lower bound in Theorem 6 suggests that no mechanism can accommodate higher abstention and sybil rates than our mechanism. Below we introduce an additional assumption that allows for a better safety-liveness tradeoff.

In particular, above we took an adversarial assumption not just w.r.t the sybil votes, but also w.r.t the passive honest votes. A less extreme approach that might be more realistic is that the active honest voters are selected uniformly at random from the honest population. As a result, we have that the votes of the active and the passive honest voters are similarly distributed.

We first consider a nonatomic population of voters, which can be thought of as the limit case of a large population. In this case, we only care about the *fraction* of voters

for each alternative, and we can assume this fraction is exactly the same among passive and active honest voters. We can see this in Figure 5, where the distribution of honest voters (in blue) under random participation is much more balanced than under arbitrary participation. This will allow us to show an improved safety-liveness tradeoff.

Later, we will extend these bounds to finite populations. This requires a probabilistic extension of the safety and liveness properties. The proofs are similar and reach the same bounds in the limit, only they use a probabilistic argument (based on the Hoeffding bound).

Theorem 7. *For a nonatomic population with random participation, the τ -RE-MJ⁺ voting rule is α -safe w.r.t Majority as the base rule if and only if $\alpha \geq \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)}$.*

Proof. Recall we denote by u_r, u_p the fraction of voters for r and p , respectively, in a voter set U .

Suppose first that $h_p > h_r - 2\alpha$. This means that there is a profile H' where we move only αh voters from r to p , and $MJ(H') = p$. Thus $p \in \overline{MJ}_\alpha(H) \subseteq B(r; \overline{MJ}_\alpha(H))$, which means α -safety holds.

Therefore, assume that $h_p \leq h_r - 2\alpha$. Intuitively, this means that the gap $h_r - h_p$ is large, and thus the gap $h_r^+ - h_p^+$ must also be small (see also Fig. 5(b)). This is the main difference from the arbitrary participation case in Fig. 5(a). The remainder of the proof is for showing that this gap is larger than $|S| - |Q| = \sigma - \tau(1 - \mu)$ (see also in Fig. 5(b)), and hence r has more active votes overall, and safety is not violated. We now turn to prove this formally.

$$\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} \leq \alpha \leq \frac{h_r - h_p}{2}. \quad (3)$$

The fraction of active voters among H is denoted by $\phi := \frac{|H^+|}{|H|} = \frac{1 - \mu - \sigma}{1 - \sigma}$.

Therefore:

$$h_r^+ - h_p^+ = \phi(h_r - h_p) \quad (4)$$

By definition, τ -RE-MJ⁺(V) = $MJ(H^+ \cup S \cup Q)$ where Q contains $\tau(1 - \mu)$ voters for r .

Thus the total fraction of active r voters is at least $h_r^+ + \tau(1 - \mu)$ (see Fig. 3(b)). As in the previous proofs, w.l.o.g. all sybils vote for p as this is the worst case for safety. We get that

$$\begin{aligned} v_r^+ - v_p^+ &\geq (h_r^+ + \tau(1 - \mu)) - (h_p^+ + \sigma) && \text{(equality when } s_p = \sigma) \\ &= (h_r^+ - h_p^+) - (\sigma - \tau(1 - \mu)) \\ &= \phi(h_r - h_p) - (\sigma - \tau(1 - \mu)) && \text{(by Eq. (4))} \\ &> \frac{1 - \mu - \sigma}{1 - \sigma} \left(\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{1 - \mu - \sigma} \right) - (\sigma - \tau(1 - \mu)) && \text{(by Eq. (3))} \\ &= (\sigma - \tau(1 - \mu)) - (\sigma - \tau(1 - \mu)) \\ &= 0, \end{aligned}$$

as required.

In the other direction (i.e. to show tightness of the bound), consider any profile where all sybils vote p , and we set h_p such that Eq. (3) holds with reversed inequality. That is,

$$\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} > \frac{h_r - h_p}{2}. \quad (5)$$

Then the inequalities in the last block of equations are reversed and we get that $v_r^+ - v_p^+ < 0$, meaning $\tau\text{-RE-MJ}^+(V) = p$.

On the other hand, for any $\alpha > \frac{h_r - h_p}{2}$, we have that $\overline{MJ}_\alpha(H) = \{r\}$ (this is exactly as in the proof of Thm. 3).

Joining both observations, $\tau\text{-RE-MJ}^+$ is not α -safe for any value of α in the range $(\frac{h_r - h_p}{2}, \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)})$. \square

Random participation does not allow us to improve the bound on liveness beyond Theorem 4, which is still tight.

As a result of Theorem 7, we get a better safety-liveness tradeoff under random participation than in Cor. 5:

Corollary 8. *Under a nonatomic population with random participation:*

- $\tau\text{-RE-MJ}^+$ is 0-safe w.r.t MJ iff $\tau \geq \frac{\sigma}{1 - \mu}$.
- $\tau\text{-RE-MJ}^+$ is 1-live iff $\tau < \frac{2(1 - \sigma - \mu)}{1 - \mu} - 1$.
- We can get both if $3\sigma + \mu < 1$.

As with arbitrary participation, we show that the $\tau\text{-RE-MJ}$ mechanism obtains the best possible tradeoff.

Theorem 9 (Lower bound for random participation). *Under random participation and nonatomic population, there is no rule \mathcal{R} such that \mathcal{R}^+ is both 0-safe and 1-live when $3\sigma + \mu \geq 1$.*

Proof. We denote by $\phi := \frac{|H^+|}{|H|} = \frac{1 - \mu - \sigma}{1 - \mu}$ the fraction of active honest voters.

Suppose the mechanism is 1-live. By 1-liveness, there is a profile V s.t. all sybils are voting for r , and $\mathcal{R}^+(V) = \mathcal{R}(S \cup H^+) = p$.

For a nonatomic population, $h_p^+ = \phi h_p$ exactly.

Now, consider a profile $\bar{V} = \bar{S} \cup \bar{H}^+ \cup \bar{H}^-$, where $|\bar{S}| = |S|$, $|\bar{H}^+| = |H^+|$, $|\bar{H}^-| = |H^-|$, so σ and μ are the same as in V . As in the proof of Thm 6, set $\bar{s}_p := \min\{h_p^+, \sigma\}$ sybils to vote for p . The difference from Thm. 6 is that we cannot set \bar{h}_p^+ directly (since they are selected at random), only \bar{h}_p . We set

$$\bar{h}_p := h_p - \frac{\bar{s}_p}{\phi}. \quad (6)$$

All other voters vote for r .

Now, note that the total amount of active p voters is

$$\bar{v}_p^+ = \bar{s}_p + \bar{h}_p^+ = \bar{s}_p + \phi \bar{h}_p = \bar{s}_p + \phi(h_p - \frac{\bar{s}_p}{\phi}) = \bar{s}_p - \bar{s}_p + \phi h_p = \phi h_p = h_p^+ = v_p^+.$$

This means that (as in Thm. 6), profiles V and \bar{V} are indistinguishable, and $\mathcal{R}^+(\bar{V}) = \mathcal{R}^+(V) = p$.

We still need to show that $\bar{h}_r \geq \bar{h}_p$, which entails a violation of 0-safety. Assume first that $\bar{s}_p < h_p^+$. Then $\bar{s}_p = \sigma$ and:

$$\begin{aligned} \phi(\bar{h}_r - \bar{h}_p) &= -\phi(\bar{h} - 2\bar{h}_p) = \phi(1 - \sigma - 2\bar{h}_p) \\ &= \phi(1 - \sigma - 2(h_p - \frac{\bar{s}_p}{\phi})) = \phi(1 - \sigma) - 2\phi h_p + 2\bar{s}_p \quad (\text{By Eq. 6}) \\ &= \frac{1 - \sigma - \mu}{1 - \sigma}(1 - \sigma) - 2h_p^+ + 2\bar{s}_p \quad (\text{By def. of } \phi) \\ &= 1 - \sigma - \mu + 2\sigma - 2h_p^+ = 1 + \sigma - \mu - 2h_p^+ \quad (\text{as } \bar{s}_p = \sigma) \\ &\geq 1 + \sigma - \mu - 2h^+ \geq 1 + \sigma - \mu - 2(1 - \sigma - \mu) \\ &= 3\sigma + \mu - 1 \geq 0, \end{aligned}$$

where the last inequality is by the premise of the theorem. Since $\phi > 0$, this entails $\bar{h}_r - \bar{h}_p \geq 0$ as well.

If $\bar{s}_p = h_p^+$ then

$$\bar{h}_p = h_p - \frac{\bar{s}_p}{\phi} = h_p - \frac{h_p^+}{\phi} = h_p - h_p = 0,$$

meaning that in \bar{V} there are no honest voters for p . In particular $\bar{h}_r > 0 = \bar{h}_p$. \square

3.3 Random Partial Participation—Finite Population

Under a finite population and random participation, we do not get a deterministic outcome, and thus the previous definitions of safety and liveness are inadequate.

We next extend the definitions to allow randomness. It does not matter if the randomness is due to voters' actions (i.e., randomly deciding whether to vote), due to the voting rule \mathcal{R} , or from other sources. We do assume however that the base rule \mathcal{G} is deterministic.

We denote by $n^+ := |H^+| = |V| \cdot h^+$ the number of active honest voters. Safety w.h.p. means that if this number is sufficiently large then the probability of an “unsafe” outcome is negligible.

Definition 8 (Safety w.h.p.). An aggregation rule \mathcal{R} is α -safe with high probability (w.r.t the base rule \mathcal{G}), if for any $V = H^+ \cup H^- \cup S$ and any $\alpha' > \alpha$, it holds that

$$Pr_{x \sim \mathcal{R}(V)}[x \notin B(r; \bar{\mathcal{G}}_{\alpha'}(H))] < \exp(-\Omega(n^+)).$$

We could similarly define liveness w.h.p., and this would make sense for various sources of uncertainty, but for our particular model this is not required: since there are

exactly $(1 - \mu - \sigma)|V|$ active honest voters, and since in the worst case for liveness, all voters vote for r , all realizations are identical. The probability that there is a violation of liveness is thus either 0 or 1.

We are now ready to show that the same upper bounds from the nonatomic case (Section 3.2) apply in the finite case as well.

Theorem 10. *For a finite population with random participation, the τ -RE-MJ⁺ voting rule is α -safe with high probability w.r.t. Majority, iff $\alpha \geq \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)}$.*

Proof. Recall we denote by u_r, u_p the fraction of voters for r and p , respectively, in a voter set U .

The proof is very similar to the proof of the nonatomic case (Thm. 7). Consider any $\alpha' > \alpha$. The case where $h_p > h_r - 2\alpha'$ is identical: it entails

$$p \in \overline{MJ}_{\alpha'}(H) \subseteq B(r; \overline{MJ}_{\alpha'}(H)),$$

which means α -safety holds regardless of the realization of active voters.

In the case that $h_p \leq h_r - 2\alpha'$, the proof is still similar, except that we augment Eq. (3) by c due to the premise of the theorem. More importantly, Eq. (4) no longer holds but we will use the Hoeffding inequality to show that it *approximately holds* with high probability. The rest of the proof is putting the technical pieces together.

To show safety w.h.p., we need to consider the case $p \notin B(r; \overline{MJ}_{\alpha'}(H))$, and upper-bound the probability that τ -RE-MJ⁺ will select p .

Denote $c := \alpha' - \alpha > 0$. Since $h_p \leq h_r - 2\alpha$, and by the premise of the theorem, we have:

$$\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} + c \leq \alpha + c = \alpha' \leq \frac{h_r - h_p}{2}. \quad (7)$$

Every honest worker is active with probability $\phi := \frac{|H^+|}{|H|} = \frac{1 - \mu - \sigma}{1 - \sigma}$. Alternatively, every active honest voter is a p voter with probability $\psi := |H_p|/|H|$.

We sample $n^+ = |H^+|$ active voters from the set $H = H_p \cup H_r$, *without replacement*. Consider n^+ samples $X_1, \dots, X_{n^+} \in \{0, 1\}$ where $X_i = 1$ if the i 'th active agent is a p voter, and 0 otherwise. Thus $n_p^+ := |H_p^+| = \sum_{i \leq n^+} X_i$ and $n_r^+ = n^+ - n_p^+$.

Observe that n_p^+ is a random variable, whose expected value is

$$n^+ \cdot E[X_i] = n^+ \psi = |H^+| \frac{|H_p|}{|H|} = \frac{|H^+|}{|H|} |H_p| = \phi |H_p|.$$

Recall that $c = \alpha' - \alpha$ and let $\varepsilon \in (0, \frac{c}{2 - 2\sigma})$. Denote the event $[n_p^+ < (\psi + \varepsilon)n^+]$ by I . By applying Hoeffding inequality,⁴

$$Pr[-I] = Pr[n_p^+ \geq (\psi + \varepsilon)n^+] < \exp(-2\varepsilon^2 n^+) = \exp(-c^2 n^+ / (1 - \sigma)^2) = \exp(-\Omega(n^+)).$$

It thus remains to show that whenever I occurs, r is selected.

⁴The Hoeffding inequality applies for sampling either with or without replacement. Without replacement it is possible to get somewhat better bounds [Serfling, 1974] but this is immaterial for our argument.

For the remainder of the proof, suppose that event I occurs, thus $n_p^+ < (\psi + \varepsilon)n^+ = \phi|H_p| + \varepsilon n^+$, and $n_r^+ = n^+ - n_p^+ > (1 - \psi - \varepsilon)n^+ = \phi|H_r| - \varepsilon n^+$ (intuitively, n_p^+, n_r^+ are close to their expected values). Therefore:

$$h_r^+ - h_p^+ = \frac{1}{n}(n_r^+ - n_p^+) > \frac{1}{n}(\phi(|H_r| - |H_p|) - 2\varepsilon n^+) = \phi(h_r - h_p) - 2\varepsilon(1 - \sigma - \mu). \quad (8)$$

By definition, τ -RE-MJ $^+(V) = MJ(H^+ \cup S \cup Q)$ where Q contains $\tau|V^+| = \tau(1 - \mu)n$ voters for r .

Thus the total fraction of active r voters is at least $h_r^+ + \tau(1 - \mu)$. As in the previous proofs, w.l.o.g. all sybils vote for p as this is the worst case for safety. We get that

$$\begin{aligned} v_r^+ - v_p^+ &\geq (h_r^+ + \tau(1 - \mu)) - (h_p^+ + \sigma) \\ &= (h_r^+ - h_p^+) - (\sigma - \tau(1 - \mu)) \\ &\geq \phi(h_r - h_p) - 2\varepsilon(1 - \sigma - \mu) - (\sigma - \tau(1 - \mu)) && \text{(by Eq. (8))} \\ &> \frac{1 - \mu - \sigma}{1 - \sigma} \left(\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{1 - \mu - \sigma} + 2c \right) \\ &\quad - 2\varepsilon(1 - \sigma - \mu) - (\sigma - \tau(1 - \mu)) && \text{(by Eq. (7))} \\ &= \frac{1 - \mu - \sigma}{1 - \sigma} \cdot \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{1 - \mu - \sigma} \\ &\quad + 2(1 - \mu - \sigma) \left(\frac{c}{1 - \sigma} - \varepsilon \right) - (\sigma - \tau(1 - \mu)) \\ &= (\sigma - \tau(1 - \mu)) + 2(1 - \mu - \sigma) \left(\frac{c}{1 - \sigma} - \varepsilon \right) - (\sigma - \tau(1 - \mu)) \\ &> 0, && \text{(since } \varepsilon < \frac{c}{1 - \sigma} \text{)} \end{aligned}$$

as required.

Tightness follows from the same construction used in the nonatomic case. Then the probability of violating safety is at least 0.5 (details omitted). \square

Therefore, Corollary 8 from the nonatomic case applies for the finite case as well, with the same bounds, except that 0-safety should be replaced with 0-safety w.h.p.

4 Extensions of the Binary Domain

4.1 Multiple Alternatives

In contrast to the binary domain, where the Majority rule is the natural base rule, when $|A| > 2$ there are many reasonable voting rules in the literature. We start by extending some of our results to Plurality voting, then considering other voting rules.

Plurality We can naturally extend the τ -RE-MJ mechanism, by using the Plurality rule $\mathcal{R} = PL$. That is, the mechanism τ -RE-PL applies the Plurality rule after adding a fraction of τ voters to r .

Unfortunately, τ -RE-PL cannot be both 0-safe and 1-live (regardless of τ), even if there is full participation and only $\sigma > 0.2$ sybils. To see why, let $\varepsilon \in (0, (\sigma - 0.2)/2)$. Consider candidates $\{r, p, p'\}$ and suppose that $h_p = 0.4$ honest voters vote p , and all other voters vote p' . Thus $h_{p'} = 1 - h_p - \sigma < 1 - 0.4 - 0.2 = h_p$ and p is the truthful outcome. A 0-safe rule must select either p or r .

Since $v_{p'} = h_{p'} + \sigma = 0.6 > v_p$, we get that p' is selected, unless $\tau \geq 0.6$. In the first option, 0-safety is violated. In the latter option, 1-liveness is violated by Theorem 4.

Instead of trying to characterize exactly the (deteriorated) safety-liveness tradeoff of τ -RE-PL, we turn to study the τ -SMJ rule from [Shahaf *et al.*, 2019] (see Definition 5).

The τ -SMJ rule naturally applies to the multiple alternative domain: if some alternative p has a supermajority, it is selected, and otherwise, r is selected. It turns out that when there are more than 2 alternatives, the two mechanisms *no longer coincide*. Moreover, τ -SMJ inherits the same safety and liveness guarantees from the binary case.

Theorem 11. *The τ -SMJ⁺ voting rule is α -safe w.r.t Majority as the base rule if and only if $\alpha \geq \frac{1+\sigma-(1+2\tau)(1-\mu)}{2(1-\sigma)}$.*

Theorem 12. *The τ -SMJ⁺ voting rule is β -live if and only if $\beta > \frac{(1-\mu)(1+2\tau)}{2(1-\sigma-\mu)}$.*

Note that the bounds in the theorems are identical to the bounds for τ -RE-MJ⁺ in the binary case (Section 3.1), except that we replace τ with 2τ .

For liveness, the number of alternatives is irrelevant so the proof of the binary case immediately applies for Theorem 12.

Proof of Theorem 11. We follow the same steps as in the proof of Theorem 3: Suppose that τ -SMJ⁺ selects p , then we need to show p is α -safe by making it the honest winner. That is, we need to construct a modified profile H' where p has most votes. In fact, we will show it gets a strict majority. For this, we need to provide corresponding inequalities to Eqs. (1) and (2).

For the first, we observe that in τ -SMJ⁺(V), alternative p gets more than $(0.5 + \tau)$ of all active votes.⁵ Thus

$$\begin{aligned} h_p^+ + \sigma &\geq v_p^+ > (0.5 + \tau)v^+ = (0.5 + \tau)(1 - \mu) \Rightarrow \\ 2h_p^+ &> (1 + 2\tau)(1 - \mu) - 2\sigma. \end{aligned} \quad (9)$$

Now, set

$$\alpha' = \alpha h \geq \frac{1 + \sigma - (1 + 2\tau)(1 - \mu)}{2}. \quad (10)$$

Then, to construct H' , we move a fraction of α honest voters to p , from any other alternative (not necessarily from r). We get:

$$\begin{aligned} 2h'_p - h &= 2(h_p^+ + \alpha') - (1 - \sigma) \\ &\geq 2h_p^+ + (1 + \sigma - (1 + 2\tau)(1 - \mu)) - (1 - \sigma) && \text{(By Eq. (10))} \\ &> (1 - \sigma) - (1 - \sigma) = 0, && \text{(By Eq. (9))} \end{aligned}$$

⁵This is exactly where the proof would fail for τ -RE-PL⁺, since p can win even with a lower fraction of votes.

so $h'_p > 0.5h$, as required. \square

As an immediate corollary from Theorems 11 and 12 we get exactly the same safety-liveness tradeoff as in Cor. 5. In particular, both 0-safety and 1-liveness can be obtained if (and only if) $3\sigma + 2\mu \leq 1$.

Note that in the example above where τ -RE-PL fails (with 0.4 of voters on p and the rest on p'), using e.g. 0.15-SMJ is 0-safe, since $v_{p'} = 0.6 < 0.65 = (0.5 + \tau)$, and thus $0.15\text{-SMJ}(V) = r$.

Another feature of the τ -SMJ rule is that it may select r even if no one voted for it!

Condorcet Conservative rules Many voting rules are guided or justified by selecting the Condorcet winner, when one exists. These rules typically differ when there is no Condorcet winner, but Reality-aware Social Choice [Shapiro and Talmon, 2018] suggests a natural conservative way to resolve this conflict in a “safe” way, by selecting the status-quo r in such cases. We call this rule the *Condorcet Conservative* rule (CC).

The τ Super Condorcet Conservative rule (τ -SCC) is similar but p_i only beats p_j if it has a supermajority of $\frac{1}{2} + \tau$ of the votes.

Proposition 13. *The following hold:*

- τ -SCC⁺ has the same liveness guarantees as τ -SMJ⁺.
- Let \mathcal{G} be any Condorcet consistent rule. Then τ -SCC⁺ has the same 0-safety guarantees w.r.t. \mathcal{G} , as τ -SMJ⁺ has w.r.t. MJ.

Proof. We prove each claim separately.

Liveness: Let $\tau, \mu, \sigma, \beta \geq 0$ such that τ -SMJ⁺ is β -live. β -liveness of τ -SMJ⁺ means that from any profile (in particular when all votes are on r) it is possible to move $\beta|H|$ voters to p so that $v_p^+ > (\frac{1}{2} + \tau)v^+$.

For any p , we can then do the same in any profile of τ -SCC⁺ by placing p at the top of all moved voters. This guarantees that over $(\frac{1}{2} + \tau)v^+$ of the voters rank p at the top, and in particular more than $(\frac{1}{2} + \tau)v^+$ rank p above any other alternative.

Safety: Let $\tau, \mu, \sigma \geq 0$ such that τ -SMJ⁺ is 0-safe.

Consider any profile $V = H \cup S$ where some $p \neq r$ wins in τ -SCC⁺(V) (otherwise safety is trivial). Then we need to show that $\mathcal{G}(H) = p$.

Indeed, consider any $p' \neq p$ (including r). Since τ -SCC⁺(V) = p , we know that in the pairwise match of p vs. p' , there is a fraction of at least $(\frac{1}{2} + \tau)v^+$ voters that prefer p . By 0-safety of τ -SMJ⁺, this means that more than half of the honest voters prefer p over p' . Since this holds for all $p' \neq p$, we have that p is the Condorcet winner of H , and thus $\mathcal{G}(H) = p$. \square

The reason why a similar proof would fail for higher values of α is that there may be different sets of honest voters preferring p over p' for each p' . We show a similar phenomenon in more detail in the next subsection.

Still, an immediate implication of Prop. 13 is that the bounds of Corollaries 5 and 8 hold also for the τ -SCC⁺ rule.

4.2 Multiple referenda

Suppose that $A = \{0, 1\}^d$, where $r = \mathbf{0}$. For a baseline, we use the issue-wise Majority rule IMJ , which simply selects the majority opinion on each of the d issues (this is a *combinatorial domain* [Lang *et al.*, 2016]). Note that $IMJ(U)$ minimizes the sum of Hamming distances to all voters in U , thus maximizing the standard definition of the social welfare.

Proposition 14. *The following hold:*

- τ - IMJ^+ has the same liveness guarantees as τ - RE - MJ^+ .
- τ - IMJ^+ has the same 0-safety guarantees w.r.t. IMJ , as τ - RE - MJ^+ has w.r.t. MJ .

Proof. For an issue $j \leq d$ and voter set U , we denote by $U|_j \in \{0, 1\}^{|U|}$ the (projected) opinions of all U voters. We prove each claim separately.

Liveness: Let $\tau, \mu, \sigma, \beta \geq 0$ such that τ - RE - MJ^+ is β -live. Consider some position $p \in \{0, 1\}^d$. For any given profile $V = H \cup S$, we define a profile H' where $\beta|H|$ voters vote p . This means that *at least* $\beta \cdot h^+$ honest voters agree with p_j for every issue j . From β -liveness of τ - RE - MJ^+ that τ - RE - $MJ^+(V'|_j) = p_j$. Thus

$$\tau$$
- RE - $IMJ^+(V') = (\tau$ - RE - $IMJ^+(V')_{j \leq d} = (\tau$ - RE - $MJ^+(V'|_j))_{j \leq d} = (p_j)_{j \leq d} = p$.

Safety: Let $\tau, \mu, \sigma \geq 0$ such that τ - RE - MJ^+ is 0-safe. Suppose that τ - RE - $IMJ^+(V) = p \neq r$ (otherwise 0-safety is trivial). To show 0-safety, we need to prove $p \in B(r; \overline{IMJ}_0(H)) = B(r; IMJ(H))$ (note that this is the first nontrivial use of the ‘‘betweenness’’ notion in the paper, see Definition 2). This means showing $p_j \in \{r_j, IMJ(H)_j\}$ for all $j \leq d$.

By 0-safety of τ - RE - MJ^+ , we know that τ - RE - $MJ^+(V|_j) \in \{r_j, MJ(H|_j)\}$ for all j . To complete the proof, we observe that $p_j = \tau$ - RE - $IMJ^+(V)_j = \tau$ - RE - $MJ^+(V|_j)$ and that $\{r_j, IMJ(H)_j\} = \{r_j, MJ(H|_j)\}$. \square

As with the Condorcet Conservative rule, we can conclude that the bounds in Corollaries 5 and 8 apply to τ - RE - IMJ .

Also as we saw for the τ - SCC rule, the τ - RE - IMJ rule does not inherit the safety properties for $\alpha > 0$.

Example 6. Suppose that $|H| = 60, |S| = 21$ (i.e. $\sigma \cong 1/4$), $\tau = 0, \mu = 0$. Then by Thm. 3 we get $1/6$ -safety of the τ - RE - MJ rule (indeed, if there are 40 honest voters on ‘0’ and 20 on ‘1’, then moving 10 = $|H|/6$ to ‘1’ is sufficient).

Now consider $A = \{0, 1\}^3$, where honest voters are dispersed as follows: 20 on $(0, 0, 1)$; 20 on $(0, 1, 0)$; 20 on $(1, 0, 0)$ and all of S are on $(1, 1, 1)$ so the outcome is $IMJ(V) = (1, 1, 1)$. However it is not possible to get $IMJ(H') = (1, 1, 1)$ by moving only 10 honest voters, since each voter can only get closer on two of the three dimensions. The best we can do is moving 5 agents from each location to $(1, 1, 1)$ (so $\alpha = 15/60$). This entails that τ - RE - IMJ is only $1/4$ -safe w.r.t. IMJ .

5 Interval Domain with Status Quo

In this section we consider voters that pick a position on a line. The natural base rule to consider here is the *Median rule (MD)*, which returns the position of the median voter on the interval. The median rule has many desired properties such as Condorcet consistency, strategyproofness, and social optimality [Black, 1948; Moulin, 1980; Procaccia and Tennenholtz, 2009].

Cohensius et al. [2017] consider the case with very few active participants but with no sybils, and we return to their model in Section 5.3. To deal with sybils, Shahaf et al. [2019] define the τ -Suppress-outer-votes median (τ -SOM) rule:⁶

- Compute the population median $m := MD(V)$.
- If $m > r$ [respectively, $m < r$], eliminate the τ fraction of voters with the highest [resp., lowest] votes. Denote the new set V^- .
- Compute the new median $m^- := MD(V^-)$.
- If $sign(m^- - r) = sign(m - r)$ then return m^- .
- Otherwise return r .

Intuitively, the τ -SOM rule first computes the median, then recompute the median after removing the extreme voters that push the median away from r . The mechanism also utilizes r to break ties, by adding a single vote for r in case of even number of votes.

5.1 RE Median versus SO Median

With full participation, applying Definition 7 to the median rule MD , we get the *reality enforcing median rule τ -RE-MD*.

Observation 15. *The τ -RE-MD mechanism and the τ -SOM rule of Shahaf et al. [2019] coincide for any τ .*

Proof. Sort voters in increasing order, and suppose, w.l.o.g (from symmetry), that $MD(V) > r$. The σ -SOM rule returns the location of the $\frac{n-\tau|V|}{2}$ voter from V . The τ -RE-MD rule returns the location of the $\frac{n+\tau|V|}{2}$ voter from $V \cup Q$. If $\frac{n-\tau|V|}{2} \leq r$, then both rules return r . Otherwise, note that the $\frac{n+\tau|V|}{2}$ voter in $V \cup Q$ is the $\frac{n+\tau|V|}{2} - \sigma|V| = \frac{n-\tau|V|}{2}$ voter in V . \square

Shahaf et al. [2019] show that the τ -SOM rule with $\tau = \sigma$ is both 0-safe and 1-live for $\sigma < \frac{1}{3}$. Next we extend this result to arbitrary values of $\tau, \sigma, \mu, \alpha$ and β , to provide an exact picture of the safety-liveness tradeoff of the rule, also for the case of partial participation.

Moreover, rather than providing an elaborate proof, we do so by a simple reduction to our results from the previous section.

⁶In [Shahaf et al., 2019] there is only one parameter $\tau = \sigma$, but the definition of the mechanism remains the same if we separate the parameters of the mechanism (τ) and the population (σ).

5.2 Reduction to the Binary Setting

We consider an arbitrary population $V = H^+ \cup H^- \cup S$ with partial participation and sybils and consider τ -RE-MD⁺. We use the following straightforward connection between the median and majority rules (proof is immediate).

Lemma 16. *Let z be the median of V , and let $x > y \geq z$. Then y has a majority against x .*

The lemma clearly still holds if we modify the set of voters by adding votes for r and/or ignoring passive voters (as long as we apply the same modification to both rules). Thus, the lemma still applies if we replace “median” with τ -RE-MD or τ -RE-MD⁺, and “majority” with τ -RE-MJ or τ -RE-MJ⁺, respectively. We use Lemma 16 to derive the following.

Theorem 17. *The following hold:*

- τ -RE-MD⁺ has the same liveness guarantees as τ -RE-MJ⁺.
- τ -RE-MD⁺ has the same safety guarantees w.r.t. MD, as τ -RE-MJ⁺ has w.r.t. MJ.

Proof. We first show that safety/liveness of τ -RE-MJ⁺ implies the same for τ -RE-MD⁺. For a profile U of locations on \mathbb{R} and a pair of locations $x, y \in \mathbb{R}$, we denote by $U|_{xy}$ the projection of U on $A = \{x, y\}$. That is, a binary profile where each voter votes for the closer alternative among x (which is considered as the status quo) and y . In case of a tie, the voter selects x . The proofs for liveness and safety are remarkably similar.

Liveness: Consider any set of parameters $\mu, \sigma, \tau, \beta \geq 0$ such that τ -RE-MD⁺ is β -live. We need to show that $\overline{\tau$ -RE-MD⁺ _{β} (V) = $(-\infty, +\infty)$. We show it includes $[r, +\infty)$, the other side is symmetric. Denote $y = \sup(\overline{\tau$ -RE-MD⁺ _{β} (V)), and assume towards a contradiction that $y < +\infty$. Then $y = \max(\overline{\tau$ -RE-MD⁺ _{β} (V)) since it is obtained as $y = \tau$ -RE-MD⁺(V') by placing all moved voters in V' on y . Let b be an arbitrary position $b > y$. For any V' obtained from V by moving a fraction of at most β voters, τ -RE-MD⁺(V') $\leq y < b$. By Lemma 16, τ -RE-MJ⁺($V'|_{yb}$) = y . Thus $\overline{\tau$ -RE-MJ⁺ _{β} ($V|_{yb}$) = $\{y\}$. Since it does not contain b , this contradicts β -liveness of τ -RE-MJ⁺ on $A = \{y, b\}$.

Safety: Consider any set of parameters $\mu, \sigma, \tau, \alpha \geq 0$ such that τ -RE-MD⁺ is α -safe. Denote $a := \sup(\overline{MD}_\alpha(H))$. If $a = +\infty$ then no violation of safety is possible and we are done. Otherwise, $a = \max(\overline{MD}_\alpha(H))$ since it is obtained as $a = MD(H')$ by placing αh honest voters on a in H' .

Denote $z := \tau$ -RE-MD⁺(V). We need to show that $z \in B(r; a) = [r, a]$. Assume otherwise towards a contradiction, i.e. that $z > a$. By Lemma 16, we have τ -RE-MJ⁺($V|_{az}$) = z . On the other hand, for any H' obtained from H by moving $\alpha|H|$ voters, we have $MD(H') \leq a < z$ and thus by the same lemma, $MJ(H'|_{az}) = a$. This entails

$$B(a; \overline{MJ}_\alpha(H|_{az})) = B(a; \{a\}) = \{a\},$$

and thus τ -RE-MJ⁺($V|_{az}$) = $z \notin B(a; \overline{MJ}_\alpha(H|_{az}))$, which is a contradiction to α -safety of τ -RE-MJ⁺.

In the other direction, if τ -RE-MJ⁺ violates safety/liveness for these parameters in some profile V , create an instance where all voters are located either on r or on p (according to their preference in V). Then, τ -RE-MD(V) = τ -RE-MJ(V) so we get a violation of safety/liveness in τ -RE-MD as well. \square

The above reduction allows us to easily transfer all previous results to the real line domain.

Corollary 18. *The following hold:*

- *Under arbitrary participation, τ -RE-MD⁺ is α -safe w.r.t MD as the base rule if and only if $\alpha \geq \frac{1+\sigma-(1+\tau)(1-\mu)}{2(1-\sigma)}$.*
- *Under arbitrary participation, there is no mechanism \mathcal{R} such that \mathcal{R}^+ is both 0-safe w.r.t MD and 1-live when $3\sigma + 2\mu \geq 1$.*
- *Under random participation and nonatomic [resp., finite] population, the τ -RE-MD⁺ voting rule is α -safe [w.h.p.] w.r.t MD as the base rule, if and only if $\alpha \geq \frac{(\sigma-\tau(1-\mu))(1-\sigma)}{2(1-\mu-\sigma)}$.*
- *Under random participation and nonatomic population, there is no mechanism \mathcal{R} such that \mathcal{R}^+ is both 0-safe w.r.t MD and 1-live when $3\sigma + \mu \geq 1$.*

5.3 Voting with Delegation

While the results above allow for partial participation, they also imply that to obtain both safety and liveness, the fraction of passive voters cannot be too large; this might be problematic in some situations. As our lower bound means that this is unavoidable, we therefore wish to relax the model to analyze other possibilities; in particular, we adopt the standard model of *proxy voting*, where only a small number of voters are active, and any passive voter delegates her vote to the nearest active voter [Alger, 2006; Cohensius *et al.*, 2017] (indeed, in certain cases this might be unrealistic, however we investigate it to better understand the landscape affecting safety and liveness).

Remark 3 (Binary votes). There is no reason in doing a separate analysis for delegation in the binary (or any categorical) domain, as, in this domain there is no difference between delegating to a proxy and actively voting (provided that every alternative has at least one active voter); sybils may still interfere, but the safety-liveness tradeoff of Majority with proxy delegation is just as in Cor. 5 with full participation ($\mu = 0$); In contrast, the position of the vote and the position of the delegatee on a line might differ.

Median with Delegation For a finite population U and a vector of vote weights $\vec{w} = (w_i)_{i \in U}$, we denote by $MD(U; \vec{w})$ the *weighted median*, where each $i \in U$ has weight $w_i \in \mathbb{N}$. Formally,

$$MD(U; \vec{w}) := \min\{u_i : i \in U, \sum_{j \leq i} w_j \geq \sum_{j > i} w_j\}.$$

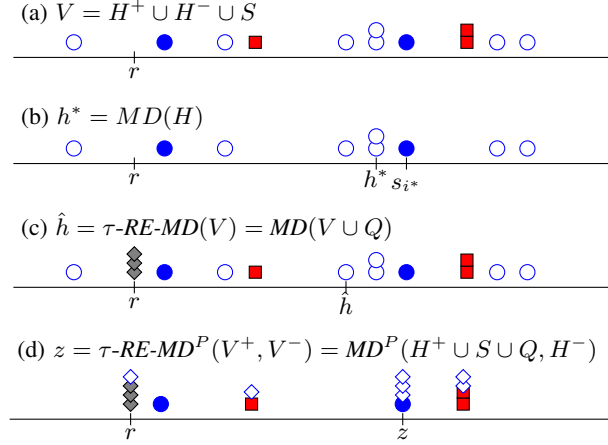


Figure 6: A demonstration of several definitions used in the proof of Theorem 19 on an example profile. The honest voters are blue circles (filled circles are active voters H^+). Sybils are marked by red squares. The gray diamonds are the virtual voters Q added by the mechanism. In the bottom figure, blue diamonds mark the followers of each active voter (their real positions are as in Fig. (c)).

We denote $n^+ := |H^+| = (1 - \mu - \sigma)|V|$. As we will see later, the fraction of active voters itself will not matter and can be arbitrarily close to 0. As in Section 3.3, we assume that active voters are sampled uniformly at random from H .

The votes of passive voters affect the outcome indirectly via delegation: for each $i \in V^+$, let $w_i = 1 + |\{j \in V^- : i = \operatorname{argmin}_{i' \in V^+} |s_i - s_{j'}|\}|$ be the number of voters for which i is the closest active voter (their “proxy”). Indeed, this follows from our strong assumption, namely that passive votes are always delegated to the closest active voter. We leave the study of alternative delegation models for future research.

The rule MD^P (P for Proxy) accepts as input two sets – active and passive voters – and returns

$$MD^P(V) = MD^P(V^+, V^-) := MD(V^+; \vec{w}),$$

where weights are set as above, according to the number of “followers” (i.e., delegates) of each $i \in V^+ \cup \{r\}$.

Recall that $MD(V)$ sees all voters in V , passive or active. $MD^+(V)$ ignores the passive voters altogether, and MD^P has no access to the ideal points of the passive voters, but knows their delegations.

As is the case throughout the paper, the Reality-Enforcing version of the median rule adds a set of virtual voters Q for the status-quo r . In contrast to the previous sections, however, the size of the input to $\tau\text{-RE-MD}^P$ is $|V|$ rather than $|V^+|$ (the mechanism knows the amount of passive voters who delegate their votes). Therefore, $\tau\text{-RE-MD}^P(V)$ returns the proxy-weighted median with additional $\tau|V|$ voters for r :

$$\tau\text{-RE-MD}^P(V) = MD^P(V \cup Q) = MD(V^+ \cup \{r\}; (\vec{w}, \tau|V|)).$$

Remark 4. If, for some passive voter i , the status-quo r is closer than all active voters, then we assume that i delegates to r (see, e.g., Fig. 6(d)).

Analysis The main theorem of this section is as follows.

Theorem 19. *Suppose that active agents are sampled uniformly at random from H . Then, the following hold:*

- Mechanism τ -RE-MD^P is $\max\{0, \frac{\sigma-\tau}{2(1-\sigma)}\}$ -safe w.h.p. w.r.t. MD;
- Mechanism τ -RE-MD^P is $\frac{1+\tau}{2(1-\sigma)}$ -live.

In the remainder of this section we prove the theorem for the special case of $\tau \geq \sigma$, which guarantees 0-safety w.h.p. The proof of the case $\tau < \sigma$ is in the appendix.

We assume, w.l.o.g, that the honest median, $h^* := MD(H)$, is to the “right” of the status-quo, i.e., $h^* \geq r$ (see Fig. 6(b)). Denote the outcome of the mechanism by

$$z := \tau\text{-RE-MD}^P(V) = MD^P(H^+ \cup Q \cup S, H^-).$$

The first lemma, cited below, says that the output of our mechanism, that uses delegations, is the closest active agent to the median of the active voters.

Lemma 20 ([Cohensius *et al.*, 2017]). *For any $U = (U^+, U^-)$, it holds that $MD^P(U)$ is the closest active agent to $MD(U)$.*

We denote the closest active voter (honest or not) to h^* by $i^* \in V$, its location by s_{i^*} (see Fig. 6(b)), and the distance between them by $d^* := |s_{i^*} - h^*|$. Intuitively, we wish this distance to be small so that we could “fix” it by moving only $\alpha|H|$ honest voters.

We define an additional point $\hat{h} := \tau\text{-RE-MD}(V)$, that is, the outcome if all voters were active (see Fig. 6(c)).

Lemma 21. *If $\tau \geq \sigma$, then $z \in [r, h^* + d^*]$.*

Proof. By Lemma 20, $z = MD^P(V \cup Q)$ is the closest active voter to $\hat{h} = MD(V \cup Q)$. Moreover, since in the worst case all S are above h^* , we have that S and Q are at opposite sides of h^* but of same size (as $\tau \geq \sigma$), $\hat{h} \leq h^*$, and thus $z \in [r, h^* + d^*]$. \square

Fig. 6(d) demonstrates a tight example, where $z = s_{i^*} = h^* + d^*$.

In the general case, d^* may be arbitrarily large. For example, it is possible that all active voters are at one extreme of the interval. However, when active voters are sampled at random, this is not very likely, as we show next. Denote by k the number of passive voters between h^* and $h^* + d^*$. Both d^* and k are random variables.

Let $c \in [0, 1]$ and denote by Y_c the ‘good’ event that $k \leq c|H|$.

Lemma 22. *If H^+ is sampled uniformly at random from H , then $1 - Pr(Y_c) \leq (1 - c)^{n^+}$.*

Proof. Consider the set of $c|H|$ honest voters closest to h^* from above. The first active honest voter we sample has a probability of exactly $(1 - c)$ to miss this set. Every subsequent sample has a lower probability since we sample without repetition. So the probability there is no active voter in the set is at most $(1 - c)^{n^+}$. \square

Moreover, the next lemma means that, if indeed the good event occurs, then we hit close to the true median; i.e., moving few honest voters suffices to fix this difference.

Lemma 23. *If Y_c occurs, then $h^* + d^* \leq \max \overline{MD}_\alpha(H)$, for any $\alpha \geq c$.*

Proof. When Y_c occurs, it holds that $\frac{k}{|H|} \leq c \leq \alpha$.

Define H' to be as H , except all voters in between h^* and $h^* + d^*$ (at most $k \leq \alpha|H|$ voters) vote for $h^* + d^*$ in H' . Then $MD(H') = h^* + d^*$. \square

We put all the pieces together to prove the following.

Proof of Theorem 19 for $\tau = \sigma$. For safety w.h.p., set $c = \alpha' \in (0, 1)$. We need to bound the probability that $z \notin B(r; \overline{MD}_{\alpha'}(H))$.

Putting everything together:

$$\begin{aligned} Pr[z \notin B(r; \overline{MD}_{\alpha'}(H))] &\leq Pr[[r, h^* + d^*] \not\subseteq B(r; \overline{MD}_{\alpha'}(H))] \quad (\text{By Lemma 21}) \\ &= Pr[h^* + d^* > \max \overline{MD}_{\alpha'}(H)] \\ &\leq Pr[\neg Y_{\alpha'}] \quad (\text{By Lemma 23}) \\ &\leq (1 - \alpha')^{n^+} = \exp(-\Omega(n^+)). \quad (\text{By Lemma 22}) \end{aligned}$$

For liveness, consider an arbitrary $a \neq r$. Assume w.l.o.g that all voters (both H and S) are on r , as this is the hardest case. We argue that changing the location of $\beta = \frac{1+\tau}{2(1-\sigma)}|V|$ honest voters from r to a will change the outcome to a . Note that it does not matter whether how many passive and active voters we move, as long as we move at least one active honest voter.

Since there are only two locations, once there are active agents on both (which occurs for any $\beta > 0$), all passive voters delegate to a voter in their own location. Thus what we get is essentially a binary vote on $\{a, r\}$, with full participation and a fraction σ of sybils. We get the liveness result immediately from Thm. 4 via the reduction of Theorem 17 (for $\mu = 0$). \square

Corollary 24. *By setting $\tau = \sigma$, the τ -RE-MD^P mechanism is both 0-safe w.h.p. and 1-live, as long as $\sigma < \frac{1}{3}$.*

This shows that delegation allows us to almost completely eliminate the drawbacks of partial participation, and get the same safety level against sybils of Shahaf et al. [2019] for full participation.

6 Discussion

Motivated by governance and mutual decision mechanisms for online communities, we have considered the common situation in which representation is threatened both by the presence of sybils, and by partial participation of the honest voters. We have defined a general mechanism, τ -*RE*- \mathcal{R} , and analyzed its safety/liveness tradeoff for several social choice settings. For a fraction σ of sybils and a fraction μ of passives in the population, we showed that, for voting on one proposal against the status-quo and voting in an interval domain, the *RE* mechanism can obtain maximal safety and liveness together as long as $3\sigma + 2\mu < 1$, generalizing all previous results of Shahaf et al. [2019]. Furthermore, we showed: that the same tradeoff applies to categorical decisions and to multiple referenda; that no mechanism can do better than τ -*RE*-*MJ*; that we can be satisfied with a somewhat lower participation rate ($3\sigma + \mu < 1$) when participation is random; and that delegation allows the same level of safety with a negligible fraction of active honest voters.

We conjecture that τ -*RE*- \mathcal{R} , when applied to other metric spaces (with suitable base rules), would guarantee similar safety/liveness tradeoffs.

To set the parameter τ (the bias towards the status-quo) effectively, after deciding upon the desired tradeoff of safety and liveness, one has to estimate σ and μ in the population. While μ can be estimated quite accurately (as an election organizer may define the set of eligible voters), this is not the case for σ . The fraction of sybils can be approximated by sampling voters [Shahaf et al., 2019, Remark 2] or by techniques that upper bound σ [Poupko et al., 2019]. Note that over-estimating σ or μ always results in a mechanism that is more safe, and thus our bounds still hold.

Finally, we are currently working on more general and realistic delegation models, in particular over social networks.

Together with state-of-the-art mechanisms for identifying and eliminating sybils [Alvisi et al., 2013], our results set the foundation for reliable and practical online governance tools.

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A Proof of Theorem 19

Theorem 19. *Suppose that active agents are sampled uniformly at random from H .*

- Mechanism τ -RE-MD^P is $\max\{0, \frac{\sigma-\tau}{2(1-\sigma)}\}$ -safe w.h.p. w.r.t. MD;
- Mechanism τ -RE-MD^P is $\frac{1+\tau}{2(1-\sigma)}$ -live.

For any two positions s, s' on the real line, we denote by $J(s, s')$ the number of passive honest agents between them. This count is directional: $J(s, s') := |\{j : s < s_j \leq s'\}|$ if $s' > s$ and $J(s, s') := -J(s', s)$ otherwise. We use lexicographic tie breaking in case there are several voters at the same location.

Also, for any location s on the line we denote by $\bar{s} := \min\{s_j : j \in H^+, s_j \geq s\}$ the active honest voter that is closest from above ($\bar{s} = +\infty$ if there is no such voter), and by $\underline{s} := \max\{s_j : j \in H^+, s_j \leq s\}$ (or $-\infty$) the one closest from below.

Denote the outcome of the mechanism by

$$z := \tau\text{-RE-MD}^P(V) = MD^P(H^+ \cup Q \cup S, H^-).$$

Throughout the section, we assume, w.l.o.g, that the honest median, $h^* := MD(H)$, is to the “right” of the status quo, i.e., $h^* \geq r$. The analysis is symmetric for the opposite case.

We define an additional point $\hat{h} := \tau\text{-RE-MD}(V)$, that is, the outcome if all voters were active (see Fig. 4(c)).

We denote the closest active voter (honest or not) to h^* by $i^* \in V$, its location by s_{i^*} , and the distance between them by $d^* := |s_{i^*} - h^*|$. Note that $s_{i^*} \leq \bar{h}^*$.

Lemma 25. $z \in [r, \bar{\hat{h}}]$.

Proof. By Lemma 20, z is on the closest active voter to \hat{h} . Since $\hat{h} \geq r$ and r is considered an active voter (see Remark 2), we get $z \geq d$.

Similarly, since $\bar{\hat{h}}$ is an active voter above \hat{h} , it must hold that $z \leq \bar{\hat{h}}$. \square

Lemma 26. *Either $\hat{h} = r$ or $J(h^*, \hat{h}) \leq \frac{\sigma-\tau}{2}|V|$.*

Proof. \hat{h} and h^* are medians of similar sets, except we added S and Q to get \hat{h} . Suppose that $\hat{h} \neq r$. All added voters are at r or above so $\hat{h} > r$.

After adding Q to H^+ , we move from h^* left to a point $\hat{h}_Q := MD(H^+ \cup Q)$, s.t. $J(\hat{h}_Q, h^*) = |Q|/2$ (every two voters move the median by one voter). Then, after adding S to $H^+ \cup Q$, in the worst case we add $|S|$ voters on the right (e.g. if all of S are at $+\infty$). Thus $J(\hat{h}_Q, \hat{h}) \leq |S|/2$.

In total,

$$\begin{aligned} J(h^*, \hat{h}) &= J(h^*, \hat{h}_Q) + J(\hat{h}_Q, \hat{h}) \\ &= J(\hat{h}_Q, \hat{h}) - J(\hat{h}_Q, h^*) \\ &\leq \frac{|S| - |Q|}{2} = \frac{(\tau - \sigma)}{2}|V|. \end{aligned}$$

\square

Denote $\hat{H} := J(\hat{h}, \bar{\hat{h}})$. Note that when active agents are selected at random, \hat{H} is a random variable. Denote by Y_c the event that $|\hat{H}| \leq c|H|$.

Lemma 27. *If H^+ is sampled uniformly at random from H , then $1 - \Pr(Y_c) \leq (1 - c)^{n^+}$.*

The proof is the same as in the case $\tau \geq \sigma$ but brought here for completeness.

Proof. Consider the set of $c|H|$ honest voters closest to \hat{h} from above. The first active honest voter we sample has a probability of exactly $(1 - c)$ to miss this set. Every subsequent sample has a lower probability since we sample without repetition. So the probability there is no active voter in the set is at most $(1 - c)^{n^+}$. \square

Lemma 28. *If $\tau \leq \sigma$ and Y_c occurs, then $\bar{\hat{h}} \leq \sup\{\overline{MD}_\alpha(H)\}$, for any $\alpha \geq \frac{\sigma - \tau}{2(1 - \sigma)} + c$.*

Proof. If $\hat{h} = r$ then $\bar{\hat{h}} = r$ and we are done.

Otherwise, we apply Lemma 26 (note we flip the order of σ, τ and thus the sign).

$$\begin{aligned} J(h^*, \bar{\hat{h}}) &= J(h^*, \hat{h}) + J(\hat{h}, \bar{\hat{h}}) = J(h^*, \hat{h}) + |\hat{H}| \\ &\leq \frac{\sigma - \tau}{2}|V| + |\hat{H}| && \text{(By Lemma 26)} \\ &\leq \frac{\sigma - \tau}{2(1 - \sigma)}|H| + c|H| && \text{(since } Y_c \text{ occurs)} \\ &= \left(\frac{\sigma - \tau}{2} + c\right)|H|. \end{aligned}$$

Note that $h^* \in \overline{MD}_\alpha(H)$. Thus if $\bar{\hat{h}} \leq h^*$ (i.e. the above equation is non-positive) then we are done.

If there is a positive number of honest voters between h^* and $\bar{\hat{h}}$, there are at most $\left(\frac{\sigma - \tau}{2(1 - \sigma)} + c\right)|H|$ such voters, i.e. a fraction of at most α , then define a new honest profile H' , where all of them are placed at $+\infty$.

This entails that $MD(H') \geq \bar{\hat{h}}$ and thus $\bar{\hat{h}} \leq \max\{\overline{MD}_\alpha(H)\}$. \square

We put all the pieces together to complete the proof.

Proof of Theorem 19. The proof in the main text proves safety for $\tau \geq \sigma$, and liveness without restriction.

It is left to show the proof for safety w.h.p., in the case $\sigma \geq \tau$.

Let $\alpha' > \frac{\sigma - \tau}{2(1 - \sigma)}$ and set $c = \alpha' - \frac{\sigma - \tau}{2(1 - \sigma)} \in (0, 1)$. We need to bound the probability that $z \notin B(r; \overline{MD}_{\alpha'}(H))$. Putting everything together:

$$\begin{aligned} \Pr[z \notin B(r; \overline{MD}_{\alpha'}(H))] &\leq \Pr[[r, \bar{\hat{h}}] \not\subseteq B(r; \overline{MD}_{\alpha'}(H))] && \text{(By Lemma 25)} \\ &= \Pr[\bar{\hat{h}} > \max \overline{MD}_{\alpha'}(H)] \\ &\leq \Pr[\neg Y_c] && \text{(By Lemma 28, as } \alpha' = \frac{\sigma - \tau}{2(1 - \sigma)} + c) \\ &\leq (1 - c)^{n^+} = \exp(-\Omega(n^+)), && \text{(By Lemma 27)} \end{aligned}$$

as required. \square