Excitations of oscillations in loaded systems with internal degrees of freedom

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Abstract

We show that oscillations are excited in a complex system under the influence of the external force, if the parameters of the system experience rapid change due to the changes in its internal structure. This excitation is collision-like and does not require any phase coherence or periodicity. The change of the internal structure may be achieved by other means which may require much lower energy expenses. The mechanism suggests control over switching oscillations on and off and may be of practical use.

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A complex system subjected to an external load is a quite common phenomenon in nature, laboratory, and industry. Almost all systems experiencing the influence of external force possess internal structure which is not affected by this force directly but may change because of intrinsic dynamics of other external causes. The examples of such systems span all possible scales. A planetary magnetosphere under the influence of the solar wind is in the equilibrium (defined by the stationarity of the magnetopause position) if the time-independent incident plasma pressure is balanced by the magnetic pressure [1]. Yet the "stiffness" of the compressed magnetosphere is determined, among others,by the magnetosphere-ionosphere-atmosphere-solid earth interaction [2]. A muscle can be heuristically represented as a set of springs connected in parallel and in series, the elasticity of these "springs" being dependent on chemical processes and electrical signals in muscular fibers [3].

These systems may oscillate globally (the position of the magnetopause, the length of the muscle) near the equilibrium position (if not overdamped). There is a rapid growth of interest in possible effect of the internal variation on the excitation of global modes. Tanimoto and Um [4] suggested that interaction with dynamical atmosphere and releases of energy due to weak local earthquakes may cause persisting global earth oscillations. Kepko and Kivelson [5] find a relation between magnetospheric bursty bulk flows and low frequency Earth magnetic field oscillations. Recently, Chagelishvili et al. [6] proposed a mechanism of oscillation excitation in a one-dimensional oscillatory system, based on the rapid change of the eigenfrequency. Unfortunately, their analytical consideration was inappropriate which obscured the physics of the phenomenon and lead them to conclusions which are too restricted.

In the present paper we investigate the excitation of oscillations in complex systems due to rapid changes in internal parameters under quite general conditions. We show that such excitation (and oscillation amplification) is a quite common phenomenon and calculate its efficiency. This question may be of significant importance in applications. One can easily imagine systems where such excitation is unwanted (like building constructions). On the other hand, the possibility of generating oscillations by not changing external force but applying only weak forces (and energy) to cause local rapid changes of the system parameters is rather attractive. A simple system where both situations are possible is an LC circuit with distributed capacity and connected to a constant emf. Excitation of oscillations at the circuit frequency is unwanted when it is used in a device designed to measure electromagnetic spectra. On the other hand, such excitation would be useful in sustaining certain level of the current when weak damping is present and it is difficult to apply periodic forcing.

In what follows we consider the systems described by the following (vector) equation of motion

$$M\ddot{\mathbf{X}} = -\frac{\partial U}{\partial \mathbf{X}} + \mathbf{F}, \tag{1}$$

where $\mathbf{X}$ is the vector of external parameters of the system (position) which are subject to the external force (position of the magnetopause under the solar wind pressure, length of the loaded muscle, charge on the effective capacitor when the LC circuit is connected to a constant emf). Let also $\mathbf{x}$ be the vector of internal parameters which are affected in a different way or vary at a much small time scale for some reasons. The generalized mass matrix M and the potential U depend both on the external and $\mathbf{X}$ and internal $\mathbf{x}$ coordinates, and the external force $\mathbf{F}$ depends on time. When the equations governing the behavior of the internal coordinates $\mathbf{x}$ are not known or are too complicated we may phenomenologically describe their influence as temporal dependence of M and U. If the system parameters and external force do not vary with time, the system is in the equilibrium position $\mathbf{X}_{eq}$ which is determined by the condition $\partial U/\partial \mathbf{X}_{eq} = \mathbf{F}$. Near the equilibrium the potential can be written as

$$U = \tfrac{1}{2} K \cdot (\mathbf{X} - \mathbf{X}_m) \cdot (\mathbf{X} - \mathbf{X}_m), \tag{2}$$

where K is the stiffness matrix, so that one has

$$M\ddot{\mathbf{X}} = -K(\mathbf{X} - \mathbf{X}_m) + \mathbf{F}, \tag{3}$$

which looks exactly as a usual oscillator equation except that now mass and stiffness are matrices. Recently it was shown that the process of vortices generation in magnetohydrodynamic and shear flows [7] can be described by a special one-dimensional case of Eq. (3).

The equilibrium position is $\mathbf{X}_{eq} = \mathbf{X}_m + K^{-1}\mathbf{F}$, and the general solution of Eq. (3) near the equilibrium point is

$$\mathbf{X} = \mathbf{X}_{eq} + \sum_i a_i \hat{e}_i \sin(\omega_i t + \phi_i), \tag{4}$$

where the frequencies ω_i and unity vectors $\hat{e}_i$ are the eigenvalues and eigenvectors of the matrix $W = M^{-1}K$, respectively. Since the equilibrium is assumed to be stable, the matrix W is positively determined and all frequencies are real. a_i and ϕ_i are the amplitudes and initial phases of the normal modes.

Mostly known channels of the energy input to the oscillating system include adiabatic change of the natural frequency, resonance with the periodically changing external force, or parametric resonance (see, for example, Ref. [8]). The last two imply (quasi)periodic behavior of the natural frequency or external force and are not within the scope of the present paper where only nonperiodic changes are considered.

The motion of the system is now fully determined by M, K, $\mathbf{X}_{eq}$, and $\mathbf{F}$. If these parameters change slowly at the typical time scale of oscillations, that is, the typical time of variation is much larger than all $1/\omega_i$, the equilibrium adiabatically shifts its position, while the amplitudes follow the well-known adiabatic law $\omega_i a_i^2 \approx$ const. In particular, the system which was in the equilibrium in the beginning will remain in the equilibrium in the course of the parameter variation, so that no oscillations are excited.

In the present paper we consider the case of rapid parameter changes, where the typical time of variation $T \ll 1/\omega_i$, for all i. We show that there this, in general, results in the excitation/amplification of oscillations in the system. A numerical example of a special type of excitation in the simplest one-dimensional system was recently considered by Chagelishvili et al. [6]. Here we consider a most general case of nonadiabatic excitation of oscillations in loaded systems near the equilibrium point.

To study quantitatively the effect let us assume that all parameters vary only within the time interval $0 < t < T$, and $\omega_i T \ll 1$. It is convenient to define $W = M^{-1}K$ and $\mathbf{f} = M^{-1}\mathbf{F}$. From Eq. (3) one immediately has

$$\dot{\mathbf{X}}(T) = \int_0^T \left[\mathbf{f}(t) - W(t)\left(\mathbf{X}(t) - \mathbf{X}_m(t)\right)\right], \tag{5}$$

where the time dependence of the system parameters is explicitly shown. In what follows we denote the system parameters and variables at $t = 0$ and at $t = T$ with subscripts 1 and 2, respectively. The solutions at $t \leq 0$ and $t \geq T$ will take the form:

$$\mathbf{X}_k(t) = \mathbf{X}_{keq} + \sum_i a_{ki}\hat{e}_{ki}\sin(\omega_{ki}t + \phi_{ki}), \tag{6}$$

where $k = 1$ for $t < 0$ and $k = 2$ for $t > T$. Since $\omega_i T \ll 1$ one can neglect the difference between $\mathbf{X}_1(T)$ and $\mathbf{X}_2(T)$, so that one has

$$\mathbf{X}_{1eq} + \sum_i a_{1i}\hat{e}_{1i}\sin(\omega_{1i}T + \phi_{1i}) = \mathbf{X}_{2eq} + \sum_i a_{2i}\hat{e}_{2i}\sin(\omega_{2i}T + \phi_{2i}). \tag{7}$$

The second matching condition is obtained from Eq. (5) in the following form:

$$\dot{\mathbf{X}}_2(T) - \dot{\mathbf{X}}_1(T) = \int_0^T (\mathbf{f} - \mathbf{f}_1)dt$$
$$+ [\int_0^T (W - W_1)dt]\mathbf{X}_1(T) - \int_0^T (W\mathbf{X}_m - W_1\mathbf{X}_{1m})dt. \quad (8)$$

It is convenient to define the instantaneous equilibrium position as $\mathbf{X}_{eq} = \mathbf{X}_m + W^{-1}\mathbf{f}$. Taking into account Eq. (6) one has

$$\sum_i \omega_{2i} a_{2i} \hat{e}_{2i} \cos(\omega_{2i}T + \phi_{2i}) = \sum_i \omega_{1i} a_{1i} \hat{e}_{1i} \cos(\omega_{1i}T + \phi_{1i})$$
$$+ [\int_0^T (W - W_1)dt] \sum_i a_{1i}\hat{e}_{1i} \sin(\omega_{1i}T + \phi_{1i}) + [\int_0^T W(\mathbf{X}_{1eq} - \mathbf{X}_{eq})dt]. \quad (9)$$

Eqs. (7) and (9) allow one to find a_{2i} and ϕ_{2i} knowing the initial state.

It is easily seen that even when $a_{1i} = 0$ for all i, the oscillation amplitude in the final state is, in general, nonzero:

$$a_{2i}^2 = [\hat{e}_{2i}^*(\mathbf{X}_{2eq} - \mathbf{X}_{1eq})]^2 + \frac{1}{\omega_{2i}^2}[\int_0^T \hat{e}_{2i}^* W(\mathbf{X} - \mathbf{X}_{1eq})]^2. \quad (10)$$

Thus, the oscillation is excited due to the (a) irreversible shift of the equilibrium position (first term in Eq. (10)) and (b) reversible temporal shift of the equilibrium position with subsequent return to the same equilibrium (the second term). The nature of the energy input is different for the two mechanisms (instantaneous change of the potential energy in the first case, and work done by the external force because of the excursion of the system from the equilibrium in the second case) but in both cases the interaction is collision-like: certain amount of momentum and energy is transferred to the system in a short time interval. The physical nature of the excitation is quite different from the proposed earlier [6] quasi-parametric resonant interaction with a Fourier-component of the changing eigenfrequency. It is worth noting that although (10) depends on the change of all parameters, M, K, and $\mathbf{X}_m$, the change of the mass M along does not affect the instantaneous equilibrium position $\mathbf{X}_{eq}$ and therefore does not result in the oscillation excitation, as could be expected.

Within the chosen approximation Eq. (10) includes all possible internal perturbations of M, K, and $\mathbf{X}_m$, and external perturbations of $\mathbf{F}$, thus giving the most general description for nonadiabatic excitation of complex systems. It includes the system considered in Ref. 6 as a special one-dimensional case where only K is changed. It is instructive to rewrite Eq. (10) for this case, where

$X_m = 0$, and $f_1 = f_2$, $W = \omega^2$, and $\omega_1 = \omega_2$. Then the excited amplitude takes the following simple form:

$$a_2 = \left| \int_0^T \frac{\omega^2}{\omega_1} \left(\frac{f_1}{\omega_1^2} - \frac{f}{\omega^2} \right) dt \right|, \tag{11}$$

from which one can easily see that the excitation occurs always and not only when the eigenfrequency decreases in the perturbation (cf. Ref. 6).

In what follows for simplicity of presentation we restrict ourselves with the one-dimensional case. Multidimensional generalization is straightforward. In the one-dimensional case Eqs. (7) and (9) immediately give

$$\begin{aligned} a_2^2 &= [(X_{1eq} - X_{2eq}) + a_1 \sin(\omega_1 T + \phi_1)]^2 \\ &+ \omega_2^{-2} [a_1 \omega_1 \cos(\omega_1 T + \phi_1) + (\int_0^T (\omega^2 - \omega_1^2) dt) a_1 \sin(\omega_1 T + \phi_1) \\ &+ \int_0^T \omega^2 (X_{1eq} - X_{eq}) dt]^2. \end{aligned} \tag{12}$$

In most cases the phase ϕ_1 of the interaction beginning is unknown (unless the perturbation is carefully prepared with some definite purpose in mind). In this case one can consider the phase ϕ_1 as random and average over random distribution to obtain eventually:

$$\begin{aligned} a_2^2 &= \tfrac{1}{2} a_1^2 \left[1 + \frac{\omega_1^2}{\omega_2^2} + \frac{(\int_0^T (\omega^2 - \omega_1^2) dt)^2}{\omega_2^2} \right] \\ &+ (X_{1eq} - X_{2eq})^2 + \frac{1}{\omega_2^2} \left[\int_0^T \omega^2 (X_{1eq} - X_{eq}) dt \right]^2. \end{aligned} \tag{13}$$

Eq. (13) describes the amplification (reduction) of oscillations (first term) and excitation of oscillations due to the shift of the of the equilibrium position and to the work of the external force during temporal shift of the equilibrium.

In the particular case of perturbation where after time interval T the system parameters return to their initial values, one arrives again at

$$a_2^2 - a_1^2 = \left[\int_0^T \frac{\omega^2}{\omega_1} \left(\frac{f_1}{\omega_1^2} - \frac{f}{\omega^2} \right) dt \right]^2, \tag{14}$$

where we neglected the term containing $[\int_0^T (\omega^2 - \omega_1^2) dt]^2 / \omega_2^2 \ll 1$. Roughly speaking, during the excursion of the internal parameters from their equilibrium values the energy of oscillations increases by $\delta E = [\int_0^T \omega^2 (X_{1eq} - X_{eq}) dt]^2$ on average. If the system experiences a series of N

such rapid variations with randomly distributed phases, the total oscillation energy increase would be about $N\delta E$, without any necessity to arrange the phases or periodicity of the variations.

Another efficient method of excitation is the rapid shift from the equilibrium position with subsequent return to this position after time $\gg 1/\omega$. In this case, neglecting the last term in Eq. (14) and after some algebra one obtains

$$a_2^2 = a_1^2 \left(\frac{\omega_1^2 + \omega_2^2}{2\omega_1\omega_2} \right)^2 + (X_{2eq} - X_{1eq})^2 \left(\frac{3}{2} + \frac{\omega_1^4 + \omega_2^4}{4\omega_1^2\omega_2^2} \right), \quad (15)$$

which shows that this scenario always results in the oscillation amplification.

To conclude, we have shown that nonadiabatic changes in the system parameters and/or external force are efficient in excitation or amplification of oscillations in driven oscillatory systems under external load. Randomly distributed in time, short nonadiabatic pulses result in efficient transfer of energy to the system. The energy transfer manifests itself in continuously increasing oscillation amplitude. This amplitude increase is not restricted to the periodically repeated (coherent) perturbations, as was suggested in [6], but occurs in quite general conditions. The energy input effect is essentially nonresonant and more collision-like where additional momentum/energy are transferred to the system at the time scale much smaller than the typical timescale of variations in the system. The effect may be important for the systems whose natural frequencies may vary quickly due to the variable internal coupling. It should be emphasized that the importance of the above analysis is well beyond the consideration of simple oscillatory systems, desribed by the simple oscillatory potential in the form (2) (chosen here for convenience and simplicity of presentation), but may be applied to quite general systems. The results (qualitatively) are valid for any system capable of (generally nonlinear) oscillations near the forced equilibrium position (although quantitative calculations would require knowledge of the structure of a particular system and ability to translate it into a low-dimensional description with small number of parameters). There is a wide spectrum of such systems, from large astrophysical scales (gravitationally bound systems, planetary magnetospheres under solar wind influence) to usual human scale (muscles, constructions) and down to small scales (electric circuits). Such generation of oscillations may be unwanted in some systems, like possible excitation of internal currents in spacecraft circuits by cosmic rays. On the other hand, the essentially nonresonant generation of oscillations might be useful in experimental determination of the natural frequencies of the systems where it is difficult to apply periodic external force but where the internal structure can be changed relatively easily and rapidly. Such methods can be also used to re-excite damped oscillations without changing of the main load.

Finally, let us use a very simple model to see whether reconnection at the dayside magnetopause may be responsible for excitations of global magnetospheric oscillations. The position of the magnetopause is determined by the balance of the incident plasma pressure $n_u m_p V_u^2$ (where n_u and V_u are the solar wind density and velocity, respectively, and m_p is the proton mass) and the magnetic pressure $B^2/8\pi$. Is the magnetopause is compressed by x, the magnetohydrodynamic conservation of magnetic flux [9] predicts that the magnetic field increases as $L/(L-x)$ where L is the equilibrium standoff distance of the magnetopause. Thus, there appears the excess force of $\sim 2n_u m_p V_u^2 xA/L$, where A is the effective area. This force has to accelerate the mass of about $n_d m_p AD$, where $n_d \approx 5n_u$ is the average plasma density, and D is the distance between the shock and magnetopause. Using the typical parameters $V_u \sim 400$ km/s, $L \sim 10R_E$, and $D \sim R_E$, where $R_E \sim 6,000$ km is the Earth radius [9], one finds the typical oscillation periods of the order of ~ 10 min. On the other hand, the typical time of reconnection should be of the order d/v_n, where $d \sim 800$ km is the magnetopause width [10], while the velocity v_n may be as high as 50 km/s [11]. Thus, the typical time of reconnection ~ 10 sec and much smaller than the oscillation period. Reconnection results in the breakdown of magnetohydrodynamics and therefore reduces the "stiffness" of the magnetic field, thus effectively temporarily reducing the global oscillation frequency. Hence, the conditions of Eq. (14) are satisfied and excitation is possible. Of course, quantitative calculations require that we are able to translate the reconnection process into the change of internal parameters, so that at this stage the proposed scenario should be considered as a speculative hypothesis.

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