IS THE FREE LOCALLY CONVEX SPACE $L(X)$ NUCLEAR?

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Dedicated to María Jesús Chasco on the occasion of her birthday

Abstract. Given a class $\mathcal{P}$ of Banach spaces, a locally convex space (LCS) E is called *multi-*$\mathcal{P}$ if E can be isomorphically embedded into a product of spaces that belong to $\mathcal{P}$. We investigate the question whether the free locally convex space $L(X)$ is strongly nuclear, nuclear, Schwartz, multi-Hilbert or multi-reflexive.

If X is a Tychonoff space containing an infinite compact subset then, as it follows from the results of [1], $L(X)$ is not nuclear. We prove that for such X the free LCS $L(X)$ has the stronger property of not being multi-Hilbert. We deduce that if X is a k-space, then the following properties are equivalent: (1) $L(X)$ is strongly nuclear; (2) $L(X)$ is nuclear; (3) $L(X)$ is multi-Hilbert; (4) X is countable and discrete. On the other hand, we show that $L(X)$ is strongly nuclear for every projectively countable P-space (in particular, for every Lindelöf P-space) X.

We observe that every Schwartz LCS is multi-reflexive. It is known that if X is a k_ω-space, then $L(X)$ is a Schwartz LCS [3], hence $L(X)$ is multi-reflexive. We show that for any paracompact first countable (in particular, metrizable) space X the converse is true, so $L(X)$ is multi-reflexive if and only if X is a k_ω-space, equivalently, if X is a locally compact and σ-compact space.

1. Introduction

This version is an extended and substantially improved revision of the previous version of our paper. [1]

All topological spaces are assumed to be Tychonoff. All vector spaces are over the field of real numbers $\mathbb{R}$.

Every Tychonoff space X can be considered as a subspace of its free LCS (locally convex space) $L(X)$ characterized by the following property: every continuous mapping $f : X \to E$ to an LCS E uniquely extends to continuous linear mapping $\widehat{f} : L(X) \to E$.

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[1]The main differences between two versions are the following. We realized that c-nuclear LCS which we introduced earlier are exactly the Schwartz LCS. Therefore, several publications devoted to the study of Schwartz spaces are relevant to our current research and they have been used in the updated version of our paper. The list of references is enlarged accordingly.

Similarly, one defines the free abelian group $A(X)$ over a Tychonoff space X: X is a subspace of generators of $A(X)$ and the topology of $A(X)$ is characterized by the property that every continuous mapping $f : X \to G$ to an abelian topological group G uniquely extends to a continuous homomorphism $\widehat{f} : A(X) \to G$.

It is known that $A(X)$ naturally embeds into $L(X)$ ([24], [26]). Given a class $\mathcal{P}$ of Banach spaces, an LCS E is called *multi-$\mathcal{P}$* if E can be isomorphically embedded in a product of spaces in $\mathcal{P}$. The following result was obtained in [27]: for every Tychonoff space X the free LCS $L(X)$ can be isomorphically embedded in the product of Banach spaces of the form $l^1(\Gamma)$, in other words, $L(X)$ is multi-$\mathcal{L}^1$, where $\mathcal{L}^1$ is the class of all Banach spaces of the form $l^1(\Gamma)$. It follows that, as a topological group, $L(X)$ admits a topologically faithful unitary representation, that is, $L(X)$ is isomorphic to a subgroup of the unitary group.

Vladimir Pestov raised the question whether $L(X)$ is *multi-Hilbert*. The question was motivated by the fact that $L(X)$ is nuclear, hence multi-Hilbert, if X is a countable discrete space [22, Ch. 3, Theorem 7.4]; in that case $L(X)$ is the locally convex direct sum of countably many one-dimensional spaces, and will be denoted by φ.

In Section 2 we prove that $L(X)$ is not multi-Hilbert whenever X contains an infinite compact subset. As a consequence we deduce that if X is a k-space, then the following properties are equivalent: (1) $L(X)$ is strongly nuclear; (2) $L(X)$ is nuclear; (3) $L(X)$ is multi-Hilbert; (4) X is countable and discrete. On the other hand, we show that $L(X)$ is strongly nuclear (hence nuclear) for every projectively countable P-space (in particular, for every Lindelöf P-space) X.

Recall the definition of nuclear maps and nuclear spaces. A linear map $u : E \to B$ from an LCS E to a Banach space B is called *nuclear* if it can be written in the form

$$u(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n,$$

where $\sum |\lambda_n| < +\infty$, (f_n) is an equicontinuous sequence of linear functionals on E, and (y_n) is a bounded sequence in B. An LCS E is called *nuclear* if any continuous linear map from E to a Banach space is nuclear.

The definition of nuclear LCS and the following important permanence results about nuclear LCS are due to A. Grothendieck [13] (see also [22, Ch. 3, §7]).

- Every subspace of a nuclear LCS is nuclear.
- Every Hausdorff quotient space of a nuclear LCS is nuclear.
- The product of an arbitrary family of nuclear LCS is nuclear.
- The locally convex direct sum of a countable family of nuclear LCS is a nuclear space.
- Every nuclear LCS is multi-Hilbert.

Note also

- The class of multi-Hilbert LCS is closed under Hausdorff quotients.

Lemma 2.4 below also should be attributed to A. Grothendieck. For the definition of a *nuclear group* and a study of free abelian nuclear groups we refer the reader to [1]. Note here only that the class of nuclear groups contains all nuclear LCS.

Michael Megrelishvili asked whether $L(X)$ is *multi-reflexive*. It turns out that this problem is tightly connected with the properties of the Schwartz locally convex spaces. The notion of Schwartz LCS was introduced by A. Grothendieck in [12]. In this paper we will use the following equivalent version: an LCS E is a *Schwartz space* if every continuous linear map from E to a Banach space is compact, that is, sends a neighborhood of zero to a set with a compact closure. For other equivalent definitions of a Schwartz space see [14, Section 3.15]. Several relevant problems in the theory of Schwartz LCS were solved in [4], [5], [19], [20], [23] (our list of references does not attempt to be complete). Note that the class of all Schwartz LCS (as well as the classes of all strongly nuclear, all nuclear, all multi-Hilbert and all multi-reflexive LCS) constitutes a *variety*, i.e. is closed under the formation of subspaces (not necessarily closed), Hausdorff quotients, arbitrary products and isomorphic images (see [6]). A group version of the concept of a Schwartz space is introduced and studied in the paper [3] (see also [2]).

We observe that every Schwartz LCS is multi-reflexive. This follows, for example, from [15, 17.2.9 and 21.1.1(c)]; we give a short proof of this assertion in Theorem 3.2. It is known that $L(X)$ is a Schwartz space, hence multi-relexive, for every k_ω-space X [3, Theorem 5.2]. In Section 3 we provide an alternative proof of this fact (Theorem 3.4).

Recall that a topological space X satisfies the first axiom of countability if each point of X has a countable base of neighborhoods. We give a complete answer to the question whether $L(X)$ is multi-reflexive for paracompact spaces X satisfying the first axiom of countability; namely, if X is such a space, then the following properties are equivalent: (1) $L(X)$ is a Schwartz space; (2) $L(X)$ is multi-reflexive; (3) X is locally compact and σ-compact (= X is the union of countably many compact sets). In particular, for a metrizable space X, $L(X)$ is multi-reflexive if and only if X is locally compact space and has a countable base. We also give a short direct proof of the fact that $L(X)$ is not multi-reflexive if X is the space of irrational numbers (Example 3.14).

Our notations are standard, the reader is advised to consult with the monographs [9], [8], [22] for the notions which are not explicitly defined in the text. In the end of the article we pose several open questions.

2. Nuclear $L(X)$

Let X be a Tychonoff space containing an infinite compact subset K. Every continuous pseudometric defined on the compact space K extends to a continuous pseudometric on X, therefore, $L(K)$ can be identified by a linear topological isomorphism with a subspace of $L(X)$ [26]. Since the free abelian group $A(K)$ naturally embeds into $L(K)$, we observe that $A(K)$ is isomorphic to a topological subgroup of $L(X)$. Hence, if $L(X)$ is a nuclear LCS, then $A(K)$ is a nuclear group. However, the free abelian group $A(K)$ is nuclear if and only if a compact space K is finite [1, Theorem 7.3]. This observation leads to

Proposition 2.1. *If a Tychonoff space X contains an infinite compact subset, then $L(X)$ is not nuclear.*

Below we prove a stronger result, the proof of which requires more elaborate arguments from functional analysis.

Theorem 2.2. *If a Tychonoff space X contains an infinite compact subset, then $L(X)$ is not multi-Hilbert.*

In order to prove Theorem 2.2 we need some preparations. Let us say that a subset C of a Banach space E is an *ellipsoid* if C is the image of a closed ball in a Hilbert space H under a bounded linear mapping $H \to E$. Note that ellipsoids are weakly compact, hence closed in E.

Lemma 2.3. *There exists a Banach space E and a countable sequence $S = (x_n)$ converging to zero in E such that S is not contained in any ellipsoid in E.*

Proof. Let E be a Banach space without the approximation property. According to [10, Theorem 1.2], there is a compact subset K in E such that for any Banach space F with a basis and any injective bounded operator $T : F \to E$ the image $T(F)$ does not contain K. Then K is not contained in any ellipsoid. Indeed, if H is a Hilbert space and $A : H \to E$ is a bounded linear operator, we can write A as the composition $H \to H' \to E$, where H' is the quotient of H by the kernel of the operator A. Then H' is a Hilbert space and the operator $T : H' \to E$ is injective, hence K is not contained in the image $T(H') = A(H)$.

Now we find a countable sequence $S = (x_n)$ in E converging to zero and such that K is contained in the closed convex hull of S. If an ellipsoid contains S, then it also contains K, which is impossible. □

Note that in fact a sequence with properties stated in Lemma 2.3 can be found in any Banach space E not isomorphic to a Hilbert space. For the proof one can use, for example, the following: in every Banach space which is non-isomorphic to a Hilbert space there is a sequence F_n of finite-dimensional subspaces whose Banach-Mazur distances from the Euclidean spaces of the same dimension go to infinity [16].

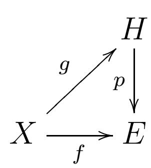
Lemma 2.4. *Let L be a multi-Hilbert LCS. Then every continuous linear map $f : L \to E$ from L to a Banach space E can be represented as a composition $f = p \circ g$, where $g : L \to H$ and $p : H \to E$ are two continuous linear maps and H is a Hilbert space.*

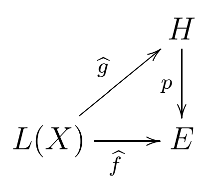
Proof. We assume that L is isomorphically embedded into the product $\prod_{i \in I} E_i$, where all E_i are Hilbert spaces. Denote by V the unit open ball of E. Since f is continuous, and by the definition of the product topology, there is a finite face $E_A = \prod_{i\in A} E_i$ of the product $\prod_{i\in I} E_i$, and a neighborhood U of zero in E_A, such that $f(p^{-1}(U))$ is contained in V. Here the set of indices $A \subset I$ is finite and $p : X \to E_A$ is the corresponding projection.

We claim that $\ker(p) \subset \ker(f)$. Indeed, if $x \in L$ and $p(x) = 0$, then $p(tx) = 0$ for every scalar t, so $tx \in p^{-1}(U))$. This means that $f(tx) = t(f(x))$ belongs to the same unit ball V for every scalar t, which implies that $f(x) = 0$. So there is some map $g : p(X) \to E$ such that $f = g \circ p$. Clearly, g is a linear map. It is also bounded, because $g(U) = f(p^{-1}(U))$ was assumed to be contained in V. The map g is defined on the vector subspace $p(X)$ of the Hilbert space E_A. (An LCS E_A is isomorphic to a Hilbert space being a finite product of Hilbert spaces). Further, the map g can be extended by continuity to the closure of that vector space in E_A. A closed vector subspace of a Hilbert space is Hilbert itself, and we get the required factorization. □

Lemma 2.5. *Let X be a non-trivial countable convergent sequence. Then $L(X)$ is not multi-Hilbert.*

Proof. On the contrary, suppose that $L(X)$ is multi-Hilbert. By Lemma 2.3 we define a 1-to-1 continuous map $f : X \to E$ such that $f(X)$ is not contained in an ellipsoid. Extend f to a linear continuous map $\widehat{f} : L(X) \to E$. Now, by Lemma 2.4, we can represent $\widehat{f}$ as a composition $\widehat{f} = p \circ \widehat{g}$, where $\widehat{g} : L(X) \to H$ and $p : H \to E$ are two continuous linear maps, $\widehat{g}$ extends the map $g : X \to H$ and H is a Hilbert space. The following diagram illustrates our construction.





The compact set $g(X) \subset H$ is contained in a ball, hence $f(X) = p(g(X))$ is contained in an ellipsoid. The obtained contradiction finishes the proof. □

Proof of Theorem 2.2. Let K be an infinite compact subset of X and assume that $L(X)$ is multi-Hilbert. As we have noted before $L(K)$ can be identified with a linear subspace of $L(X)$, hence $L(K)$ is multi-Hilbert. K is an infinite compact space, therefore we can choose a continuous map $\pi : K \to [0, 1]$ with an infinite image. Denote by $M = \pi(K)$. We have that $\pi : K \to M$ is a closed (hence, quotient) continuous surjective mapping. In this situation π lifts to a linear continuous quotient mapping $\widehat{\pi}$ from $L(K)$ onto $L(M)$. The latter fact can be proved analogously to [18, Proposition 1.8]. Since the class of multi-Hilbert LCS is closed under Hausdorff quotients we conclude that $L(M)$ is also multi-Hilbert. However, M is an infinite compact subset of the segment $[0, 1]$, therefore M contains a copy of a convergent sequence S. This would mean that $L(S)$ is multi-Hilbert contradicting Lemma 2.5. The obtained contradiction finishes the proof of Theorem 2.2. □

Recall that a topological space X is called a k-space whenever $F \subset X$ is closed iff the intersection $F \cap K$ is closed in K for every compact $K \subset X$. If X can be covered by countably many compact subsets K_n such that $F \subset X$ is closed iff all intersections $F \cap K_n$ are closed, then X is said to be a k_ω-space [11].

An LCS E is called *strongly nuclear* [19, 1.4] if for every continuous seminorm q on E there exist a rapidly decreasing sequence (λ_n) (this means that $\sum n^k |\lambda_n| < \infty$ for every k) and an equicontinuous sequence (a_n) of linear functionals on E such that $q(x) \leq \sum |\lambda_n| |\langle x, a_n \rangle|$ for all $x \in E$. It is easy to verify that $L(\mathbb{N}) = \varphi$ is strongly nuclear, where $\mathbb{N}$ denotes the discrete space of natural numbers.

Following [6] denote by $\mathcal{V}(\varphi)$ the variety of LCS generated by φ. Note that $\mathcal{V}(\varphi)$ is the (unique) second smallest variety, in the sense that every variety properly containing $\mathcal{V}(\mathbb{R})$ contains $\mathcal{V}(\varphi)$ [6].

Corollary 2.6. *Let X be a k-space. The following are equivalent*

(i) *$L(X)$ is strongly nuclear.*
(ii) *$L(X)$ is nuclear.*
(iii) *$L(X)$ is multi-Hilbert.*
(iv) *X is a countable discrete space.*

Proof. Only one implication (iii) $\Longrightarrow$ (iv) is new and needs to be proved. By Theorem 2.2, every compact subset of X is finite, therefore the k-space X is discrete. Every multi-Hilbert space is multi-reflexive, and we will show in Lemma 3.8 that $L(X)$ is not multi-reflexive if X is an uncountable discrete space. □

Besides the requirement that X should contain no infinite compact subspaces, there is another necessary condition for $L(X)$ to be nuclear.

Theorem 2.7. *Let $L(X)$ be a nuclear LCS. Then every metrizable image of X under a continuous map must be separable.*

Proof. Let $f : X \to M$ be a continuous map onto a metrizable space M. Assume that the topology of M is generated by a metric d. Denote by $B = C_b(M)$ the Banach space of all bounded continuous real-valued functions on M equipped with the supremum norm. It is known that the Kuratowski mapping isometrically embeds M into the Banach space B. Specifically, if (M, d) is a metric space, a is a fixed point in M, then the mapping $\Phi : M \to C_b(M)$ defined by

$$\Phi(x)(y) = d(x, y) - d(a, y) \quad \text{for all} \quad x, y \in M,$$

is an isometry.

Extend the map $f : X \to M$ to a linear continuous map $\widehat{f} : L(X) \to B$. It follows from the definition of a nuclear map that the range of $\widehat{f}$ is separable. Since M is contained in that range, it is separable, too. □

It follows from Theorem 2.8 below that there exist non-separable spaces X such that $L(X)$ are strongly nuclear, hence nuclear. We say that a Tychonoff space X is *projectively countable* if every metrizable image of X under a continuous map is countable. Recall that a *P-space* is a topological space in which the intersection of countably many open sets is open. To see that Lindelöf P-spaces are projectively countable, observe that if X is a P-space, all points of Y are G_δ, and $f : X \to Y$ is a continuous map, then the inverse images of points from Y form a disjoint open cover of X. It is known that ω-modification of a topology of any scattered Lindelöf space produces a Lindelöf P-space.

Theorem 2.8. *If X is a projectively countable P-space (in particular, a Lindelöf P-space), then $L(X)$ is strongly nuclear.*

Proof. Let M be a metrizable space and $f : X \to M$ be a continuous map. Every point $m \in M$ is a G_δ-set in M, therefore every fiber $f^{-1}(m)$ is open in the P-space X. We obtain that X can be decomposed into the free countable union of open fibers $f^{-1}(m), m \in M$. It follows that every continuous map $f : X \to M$ admits a factorization $X \to \mathbb{N} \to M$, where $\mathbb{N}$ is the discrete space of natural numbers, hence any linear continuous map $L(X) \to B$ to a Banach space B admits a factorization $L(X) \to L(\mathbb{N}) \to B$. Using [19, 3.4. Remark] we can conclude that $L(X)$ can be isomorphically embedded into a power of the space φ. This means that $L(X)$ belongs to $\mathcal{V}(\varphi)$ which is a sub-variety of the variety of all strongly nuclear spaces [6], since $L(\mathbb{N}) = \varphi$ is strongly nuclear. □

It turns out that there are projectively countable P-spaces which are not Lindelöf P-spaces. A Tychonoff space X is called *cellular-Lindelöf* if for every disjoint family $\mathcal{U}$ of open nonempty subsets of X there is a Lindelöf subspace L of X such that L meets every member of the family $\mathcal{U}$. It is easy to see that every cellular-Lindelöf P-space is projectively countable. Hence, Theorem 2.8 implies

Proposition 2.9. *If X is a cellular-Lindelöf P-space, then $L(X)$ is a strongly nuclear LCS.*

Another quick consequence provides sufficient conditions for the free abelian group $A(X)$ to be nuclear:

Corollary 2.10. *If X is a cellular-Lindelöf P-space, then $A(X)$ is a nuclear topological group.*

See [25, Theorem 3.16] for an example of a cellular-Lindelöf P-space which is not even weakly Lindelöf.

3. Multi-reflexive $L(X)$

Recall that an LCS is *multi-reflexive* if it is isomorphic to a subspace of a product of reflexive Banach spaces. For instance, every LCS endowed with its weak topology is multi-reflexive, since it embeds into a product of the real lines. However, a reflexive LCS need not be multi-reflexive [17, Remark 4.14 (a)].

Our proof of Theorem 2.2 is based on the fact that certain compact subsets of Banach spaces are not contained in ellipsoids. However, every weakly compact subset of a Banach space is contained in the image of the closed unit ball of a reflexive space under a bounded linear mapping (see [9, Theorem 13.22]). This motivates the following question (suggested to us by Michael Megrelishvili): is $L(X)$ multi-reflexive?

We have seen that $L(X)$ is not multi-Hilbert if X is an infinite compact space (Theorem 2.2). Nevertheless, $L(X)$ is a Schwartz LCS for every compact space X [3]. We provide an alternative proof of this statement (Theorem 3.4) and due to the fact that every Schwartz LCS is multi-reflexive (Theorem 3.2), we conclude that $L(X)$ is multi-reflexive for every compact space X (Theorem 3.5).

In general, $L(X)$ need not be Schwartz or multi-reflexive. We provide a complete characterization of first countable paracompact spaces X such that $L(X)$ is Schwartz or multi-reflexive (Theorem 3.12). In particular, if X is metrizable, then $L(X)$ is multi-reflexive iff $L(X)$ is a Schwartz LCS iff X is locally compact and has a countable base (Corollary 3.13).

Lemma 3.1. *If L is a Schwartz LCS, then every continuous linear map $L \to B$ to a Banach space B admits a factorization $L \to B_1 \to B$, where B_1 is a Banach space, $L \to B_1$ is a continuous linear map and $B_1 \to B$ is a compact operator.*

Proof. Let U be a balanced convex neighborhood of zero in L such that the image of U in B has a compact closure. Let p be the gauge functional corresponding to U. We can take for B_1 the completion $\widetilde{L}_U$ of the normed space L_U associated with the seminorm p, as in [22, Ch. 3, §7]. □

Theorem 3.2. *Every Schwartz LCS is multi-reflexive.*

Proof. Let L be a Schwartz LCS. We want to prove that continuous linear maps from L to reflexive Banach spaces generate the topology of L. Since maps of L to all Banach spaces generate the topology of L, it suffices to prove that every continuous linear map $L \to B$ to a Banach space admits a factorization $L \to B_1 \to B$, where B_1 is a reflexive Banach space. According to Lemma 3.1, we can find a factorization $L \to B_2 \to B$, where $B_2 \to B$ is a compact operator between Banach spaces. Compact operators are weakly compact, and in virtue of the Davis–Figiel–Johnson–Pełczyński theorem [9, Theorem 13.33] weakly compact operators between Banach spaces admit a factorization through reflexive spaces. Thus we have a required factorization $L \to B_2 \to B_1 \to B$ for some reflexive Banach space B_1. □

Let $L_0(X)$ be the hyperplane $\{\sum c_i x_i : \sum c_i = 0\}$ in $L(X)$. The topology of $L_0(X)$ is generated by the seminorms $\overline{d}$ of the following form. Let d be a continuous pseudometric on X. The seminorm $\overline{d}$ on $L_0(X)$ is defined by

$$\overline{d}(v) = \inf\left\{\sum |c_i| d(x_i, y_i) : v = \sum c_i(x_i - y_i),\ c_i \in \mathbb{R},\ x_i, y_i \in X\right\}.$$

Denote by B_d the Banach space associated with the seminormed space $(L_0(X), \overline{d})$, that is, the completion of the quotient by the kernel of $\overline{d}$. The dual Banach space B_d^* is naturally isomorphic to the quotient $C_d/\mathbb{R}$ of the space of d-Lipschitz functions on X by the one-dimensional subspace of constant functions. The seminorm on C_d assigns to each d-Lipschitz function f its Lipschitz constant,

$$\|f\| = \inf \{k \geq 0 : |f(x) - f(y)| \leq k \cdot d(x, y) \,\text{for all}\, x, y \in X\}\,.$$

Lemma 3.3. *If X is a compact space and d is a continuous pseudometric on X, then the natural operator $B_{\sqrt{d}} \to B_d$ is compact.*

Proof. We may assume that d is a metric. Pick a point $x_0 \in X$, and identify the quotient $C_d/\mathbb{R}$ with the Banach space $C'_d = \{f \in C_d : f(x_0) = 0\}$. A bounded linear operator between Banach spaces is compact if and only if its dual is compact, by the Shauder theorem [9, Theorem 15.3], so it suffices to prove that the embedding $C'_d \to C'_{\sqrt{d}}$ is compact. Take any sequence (f_n) in the unit ball of C'_d, that is, a sequence of d-non-expanding functions such that $f_n(x_0) = 0$. By the Arzelà–Ascoli theorem, the sequence (f_n) has a uniformly convergent susbsequence. To simplify the notation, assume that the sequence (f_n) itself uniformly converges to a limit f. We claim that (f_n) converges to f in $C'_{\sqrt{d}}$ as well. Let $\epsilon > 0$ be given. Put $g_n = f_n - f$. We must prove that for n large enough we have

$$|g_n(x) - g_n(y)| \leq \epsilon\sqrt{d(x, y)} \tag{3.1}$$

for all $x, y \in X$. Note that each g_n is 2-Lipschitz with respect to d. If x and y are close to each other, namely, if $d(x, y) \leq \epsilon^2/4$, then (3.1) holds:

$$|g_n(x) - g_n(y)| \leq 2d(x, y) = 2\sqrt{d(x, y)}\sqrt{d(x, y)} \leq \epsilon\sqrt{d(x, y)}.$$

On the other hand, for pairs x, y such that $d(x, y) > \epsilon^2/4$ the ineqality 3.1 holds for n large enough because the sequence (g_n) uniformly converges to zero. □

Theorem 3.4. *If X is a compact space, then $L(X)$ is a Schwartz LCS.*

Proof. Let $T : L(X) \to B$ be a continuous linear operator, where B is a Banach space. We want to prove that T is compact. Let d be the pseudometric on X induced by T from the metric on B. We have a factorization $L(X) \to B_d \to B$ of the operator T. By Lemma 3.3, the middle arrow in the factorization $L(X) \to B_{\sqrt{d}} \to B_d \to B$ is a compact operator. It follows that T is compact as well. □

Combining Theorems 3.2 and 3.4, we get the result we were aiming at.

Theorem 3.5. *If X is a compact space, then $L(X)$ is multi-reflexive.*

Theorems 3.4 and 3.5 can be readily generalized to the case of k_ω-spaces, similarly to [3]. Indeed, let $X = \bigcup K_n$ be a k_ω-space, where (K_n) is a sequence of compact subsets of X witnessing the k_ω-property. Then X is a quotient of the free sum $\bigoplus K_n$ [11], and $L(X)$ is a quotient of $L(\bigoplus K_n)$ which is isomorphic to the countable locally convex sum of $L(K_n)$. Since Schwartz spaces are preserved by quotients and by locally convex sums of countable sequences [15, Proposition 21.1.7], we conclude that $L(X)$ is a Schwartz LCS.

In particular, we have established the following (which is a somewhat special case of [3]):

Theorem 3.6. *If X is a locally compact σ-compact space, then $L(X)$ is a Schwartz LCS and hence (Theorem 3.2) multi-reflexive.*

Our next aim is to show that for a wide and natural class of Tychonoff spaces X the converse of Theorem 3.6 is true: $L(X)$ is a Schwartz LCS if and only if X is a locally compact and σ-compact space.

Lemma 3.7. *If $L(X)$ is multi-reflexive, then for every Banach space E and every continuous map $f : X \to E$ the image $f(X)$ can be covered by countably many weakly compact subspaces of E.*

Proof. If $L(X)$ is multi-reflexive, by literally the same arguments as we have used in the proof of Lemma 2.4, we can represent f as a composition $f = p \circ g$, where $g : X \to F$ is a continuous map to a reflexive Banach space F and $p : F \to E$ is a bounded linear map. Since the reflexive Banach space F is the union of countably many weakly compact sets, the same is true for the set $p(F)$ which is a subset of E containing $f(X)$. □

Lemma 3.8. *If X is an uncountable discrete space, then $L(X)$ is not multi-reflexive.*

Proof. Consider the Banach space $E = l^1(\Gamma)$, where Γ is a set of indices with the same cardinality as X. Let $Y = \{e_\gamma : \gamma \in \Gamma\}$ be the canonical basis of E. Let f be any bijection from X onto Y, then $f : X \to E$ is a continuous map. We claim that $Y = f(X)$ cannot be covered by countably many weakly compact subspaces of E. This follows from the easily verified fact that Y is a closed discrete subset of E endowed with the weak topology (use the natural pairing $\langle (a_\gamma), (b_\gamma) \rangle = \sum a_\gamma b_\gamma$ between $E = l^1(\Gamma)$ and $E' = l^\infty(\Gamma)$, and consider 0–1 sequences in E'). Alternatively, note that $l^1(\Gamma)$ has the Schur property, that is, every weakly converging sequence in $l^1(\Gamma)$ converges in the norm topology, and then every weakly compact subset of $l^1(\Gamma)$ is compact in the norm topology (see [9, Exercise 5.47]). Hence, no infinite subset of the closed discrete set Y can be covered by a weakly compact subset of E, and the claim follows. □

The following known fact plays a crucial role in the sequel.

Theorem 3.9. [26] *Let X be a paracompact (in particular, metrizable) space. Then for every closed subspace $Y \subset X$, the natural map $L(Y) \to L(X)$ is a topological embedding.*

Corollary 3.10. *If X is a non-separable metric space, then $L(X)$ is not multi-reflexive.*

Proof. X contains an uncountable closed discrete subspace, so Lemma 3.8 and Theorem 3.9 apply. □

The *metric fan* M is the metrizable space defined as follows. As a set, M is the set $\mathbb{N} \times \mathbb{N}$ of pairs of natural numbers plus a point p "at infinity"; p is the only non-isolated point of M, and a basic neighborhood U_n of p consists of p and all pairs (a, b) such that $b > n$. Thus M is the union of countably many sequences converging to p. It is easy to verify that the space M is not locally compact.

Lemma 3.11. *If M is the metric fan defined above, then $L(M)$ is not multi-reflexive.*

Proof. Consider the separable non-reflexive Banach space $E = l^1$, and let (e_n) be the canonical basis. The collection of all vectors of the form e_n/k, $k = 1, 2, \dots$, together with the zero vector, is a subspace $A \subset E$ homeomorphic to M. Let $f : M \to A$ be a homeomorphism, $\widehat{f} : L(M) \to E$ its linear extension. We want to prove that there is no factorization $L(M) \to F \to E$ of $\widehat{f}$, where F is a reflexive Banach space. Suppose there is such a factorization. Let A' be the image of M in F, and let p be the non-isolated point of A'. Denote the arrow $F \to E$ by g. We have continuous bijections $M \to A' \to A$, and their composition is a homeomorphism. It follows that the restriction $g \restriction_{A'}: A' \to A$ is a homeomorphism. Let U be the closed unit ball in F. Then $(p + U) \cap A'$ is a neighborhood of p in A'. Since $g \restriction_{A'}: A' \to A$ is a homeomorphism, $g((p + U) \cap A')$ is a neighborhood of $g(p) = 0$ in A, and so is the bigger set $g(U) \cap A$. Thus for k large enough all the vectors e_n/k lie in the weakly compact set $g(U)$. Multiplying by k, we conclude that all the vectors e_n lie in a weakly compact subset of E. That is a contradiction: we noted in the proof of Lemma 3.8 that the set $\{e_n\}$ is closed and discrete with respect to the weak topology of E. □

We now are ready to prove one of the main results of our paper.

Theorem 3.12. *Let X be a first countable paracompact space. The following conditions are equivalent:*

(i) *$L(X)$ is a multi-reflexive LCS;*
(ii) *$L(X)$ is a Schwartz LCS;*
(iii) *X is locally compact and σ-compact.*

Proof. (iii) ⇒ (ii): this is Theorem 3.6.

(ii) ⇒ (i): this is Theorem 3.2.

(i) ⇒ (iii): By Theorem 3.9 and Lemma 3.11, X does not contain a closed copy of the metric fan M. A first countable paracompact space is locally compact if and only if it does not contain a closed subspace homeomorphic to M (by [7, Lemma 8.3]). Thus X is locally compact. Being locally compact and paracompact, X can be represented as a disjoint union of clopen σ-compact subspaces (by [8, Theorem 5.1.27]). Since X does not contain an uncountable closed discrete subspace (by Lemma 3.8 and Theorem

3.9), the number of components in the representation of X above is countable and the proof is complete. □

Corollary 3.13. *Let X be a metrizable space. The following conditions are equivalent:*

(i) *$L(X)$ is a multi-reflexive LCS;*
(ii) *$L(X)$ is a Schwartz LCS;*
(iii) *X is locally compact and has a countable base.*

Example 3.14. If P is the space of irrational numbers, then as a straightforward consequence of Theorem 3.12 we have that $L(P)$ is not multi-reflexive. We note that an alternative proof of this fact readily follows from Lemma 3.7.

Indeed, consider any separable non-reflexive Banach space E. For example, we can take the separable Banach space $E = l^1$. Then weakly compact subsets of E have empty interior, and it follows from the Baire Category Theorem that E is not the union of countably many weakly compact sets. On the other hand, E, like any Polish space, is the image of P under a continuous map. Now apply Lemma 3.7.

4. Open problems

Problem 4.1. *Characterize all Tychonoff space X such that $L(X)$ are nuclear / multi-Hilbert.*

Problem 4.2. *Characterize all Tychonoff spaces X such that $L(X)$ are Schwartz / multi-reflexive.*

Problem 4.3. *Does there exist a Tychonoff space X such that $L(X)$ is multi-Hilbert but not nuclear?*

Problem 4.4. *Does there exist a Tychonoff space X such that $L(X)$ is multi-reflexive but not Schwartz?*

A more specific question is the following.

Problem 4.5. *Let p be a point from the the remainder of the Stone-Čech compactification $\beta\mathbb{N} \setminus \mathbb{N}$. Denote by $X = \mathbb{N} \cup \{p\}$. Is $L(X)$ nuclear or multi-Hilbert?*

Note that every Schwartz space can be isomorphically embedded in some sufficiently high power of the Banach space c_0 [20]. There are examples of Schwartz spaces which cannot be isomorphically embedded in any power of the Banach space l^p $(1 < p < \infty)$ [4]. On the other hand, every nuclear space can be isomorphically embedded in some sufficiently high power of every Banach space [21]. These facts provide motivation for the following questions.

Problem 4.6. *Characterize all Tychonoff spaces X such that $L(X)$ can be isomorphically embedded in some sufficiently high power of the Banach space c_0.*

Problem 4.7. *Characterize all Tychonoff spaces X such that $L(X)$ can be isomorphically embedded in some sufficiently high power of the Banach space l^p $(1 \leq p < \infty)$.*

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