A UNIFIED FRAMEWORK FOR CONTINUOUS/DISCRETE POSITIVE/BOUNDED REAL STATE-SPACE SYSTEMS

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Abstract. There are four variants of passive, linear time-invariant systems, described by rational functions: Continuous or Discrete time, Positive or Bounded real. By introducing a quadratic matrix inequality formulation, we present a unifying framework for state-space characterization (a.k.a. Kalman-Yakubovich-Popov Lemma) of the above four classes of passive systems.
These four families are matrix-convex as rational functions, and a slightly weaker version holds for the corresponding balanced, state-space realization arrays.

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Contents

1. Introduction

In the study of dynamical systems, passivity is a fundamental property. Thus, it has been extensively addressed in various frameworks. We here focus on finite-dimensional, linear, time-invariant, passive systems described by matrix-valued real rational functions of a complex variable z.

We shall use the following notation: $\mathbb{C}_L$ or $\mathbb{C}_R$ is the open Left or Right half of the complex planes ($\overline{\mathbb{C}}_R$ is the closed right half plane). Let also $\mathbb{D} = \{z \in \mathbb{C} \ : \ 1 > |z| \ \}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} \ : \ 1 \geq |z| \ \}$, be the open, closed unit disk and $\overline{\mathbb{D}}^c = \{z \in \mathbb{C} \ : \ |z| > 1 \ \}$ is the the exterior of the closed unit disk[1].

For simplicity of exposition we begin with scalar functions terminology:

[1]The superscript c stands for "complement".

(α) $\mathcal{P}$, Positive-Real (continuous-time) analytically mapping $\mathbb{C}_R$ to its closure, $\overline{\mathbb{C}}_R$. See e.g. [6, Chapter 5], [11, Chapter 7], [12, Subsection 2.7.2], [17]. and [28].

(β) $\mathcal{B}$, Bounded-Real, (continuous-time) analytically mapping, $\mathbb{C}_R$ to $\overline{\mathbb{D}}$. See [5], [6, Section 7.2], [11, Chapter 7], [12, Subsection 2.7.3], and [17].

(γ) $\mathcal{DP}$, Discrete-Time-Positive-Real analytically mapping $\overline{\mathbb{D}}^c$ to $\overline{\mathbb{C}}_R$, the closed right-half plane. See e.g. [21], [24], [27], [29] and [39, Lemma 1].

(δ) $\mathcal{DB}$, Discrete-Time-Bounded-Real analytically mapping $\overline{\mathbb{D}}^c$ to $\overline{\mathbb{D}}$. See e.g. [30], [32], [36] and [39, Lemma 7].

The above families, are common in Engineering circles, for example $\mathcal{P}$ and $\mathcal{DB}$ are used to describe passive systems in continuous and discrete-time set-up, respectively.

For completeness, we point out that in mathematical analysis community there are additional sets:

Herglotz or Carathéodory functions analytically map $\mathbb{D}$ to $\overline{\mathbb{C}}_R$. See [23]. In other words, if $F(z)$ is a Herglotz function, $F\left(z^{-1}\right)$ is a $\mathcal{DP}$ function.

Schur functions analytically map $\mathbb{D}$ to its closure $\overline{\mathbb{D}}$. See e.g. [16] and [34]. In other words, if $F(z)$ is a Schur function, $F\left(z^{-1}\right)$ is a $\mathcal{DB}$ function.

Recall that whenever $F(z)$ is an $p \times m$-valued rational function with no pole at infinity, i.e. $D := \lim_{z \to \infty} F(z)$ is well-defined, one can associate with it a corresponding $(n+m) \times (n+m)$ state-space realization array, R_F i.e.

$$F(z) = C(zI_n - A)^{-1}B + D \qquad R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{1.1}$$

The $(n+p) \times (n+m)$ realization R_F in Eq. (1.1) is called *minimal*, if n is the McMillan degree of $F(z)$.

In this work we focus on the case $p = m$ and examine characterizations of the above four families of passive systems through the corresponding state-space realizations. This is also known as the Kalman-Yakobovich-Popov Lemma. For a (modest) account of the vast literature on the subject, beyond those mentioned thus far, see e.g. [3], [10] [15], [22] (a survey), [25], [26], [33], [37] and [38]. For infinite-dimensional versions (all study Schur functions in the above terminology) see e.g. [8], [9], [35].

This work is organized as follows. In Section 2 we give the basic background. Matrix-convex sets are introduced the Section 3. In Section 4, the main result given and illustrated by a detailed example. In Section 5 we specialize the main result to subsets of the families of functions discussed in Section 4. In Section 6 the uniform framework is applied to show matrix-convexity of systems, in the set-up of state-space realizations.

2. Preliminary Background

In the sequel we shall denote by $(\mathbf{H}_n)$ $\overline{\mathbf{H}}_n$ the set of $n \times n$ (non-singular) Hermitian matrices. Skew-Hermitian matrices are denoted by, $i\overline{\mathbf{H}}_n$. It is common to take $\overline{\mathbf{H}}$ and $i\overline{\mathbf{H}}$ as the matricial extension of $\mathbb{R}$ and $i\mathbb{R}$, respectively. Then within Hermitian matrices $(\mathbf{P}_n)$ $\overline{\mathbf{P}}_n$ will be the respective subsets of positive (definite) semi-definite matrices. Recall that $\overline{\mathbf{P}}_n$ may be viewed as the closure of the open set $\mathbf{P}_n$.

For $H \in \mathbf{H}_n$ let us define the following sets satisfying the Lyapunov inclusion,

$$\overline{\mathbf{L}}_H := \left\{ A \in \mathbb{C}^{n\times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \overline{\mathbf{P}}_n \right\}$$

(2.1)

$$\mathbf{L}_H := \left\{ A \in \mathbb{C}^{n\times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \mathbf{P}_n \right\},$$

and the Stein inclusion,

$$\overline{\mathbf{S}}_H := \left\{ A \in \mathbb{C}^{n\times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \overline{\mathbf{P}}_n \right\}$$

(2.2)

$$\mathbf{S}_H := \left\{ A \in \mathbb{C}^{n\times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \mathbf{P}_n \right\}.$$

The above Quadratic Matrix Inclusion formulation is not the common way to describe the families $\mathbf{L}_H$ and $\mathbf{S}_H$. Yet it enables us to present these sets in a common framework. This approach will be taken a step forward in Theorem 4.5 below.

The sets $\overline{\mathbf{L}}_H$ and $\overline{\mathbf{S}}_H$ may be viewed as the closure of the open sets $\mathbf{L}_H$ and $\mathbf{S}_H$, respectively. The sets $\overline{\mathbf{L}}_H$ and $\mathbf{L}_H$ were introduced and studied in [13]. In [7] Tsuyoshi Ando *characterized* the set $\mathbf{S}_H$.

We now resort to the classical Cayley transform. Recall that $\mathcal{C}(A)$, the Cayley transform of a matrix $A \in \mathbb{C}^{n\times n}$, is given by

$$\mathcal{C}(A) := (I_n - A)(I_n + A)^{-1} = -I_n + 2(I_n + A)^{-1}, \qquad -1 \notin \text{spect}(A). \tag{2.3}$$

Recall also that this transform is involutive, i.e. whenever defined, $\mathcal{C}(\mathcal{C}(A)) = A$.

It is well known that for a given $H \in \mathbf{H}_n$,

$$\mathcal{C}\left(\overline{\mathbf{L}}_H\right) = \overline{\mathbf{S}}_H \qquad \mathcal{C}(\mathbf{L}_H) = \mathbf{S}_H. \tag{2.4}$$

When $-H \in \mathbf{P}_n$, the set $\mathbf{L}_H$ is associated with Hurwitz stability of differential equations of the form $\dot{x} = Ax$. Similarly, when $H \in \mathbf{P}_n$, the set $\mathbf{S}_H$ is associated with Schur stability of difference equations of the form $x(k+1) = Ax(k)$.

In the sequel, we shall focus on the special case where in Eqs. (2.1) and (2.2) $H = I_n$, i.e.

$$\overline{\mathbf{L}}_{I_n} := \left\{ A \in \mathbb{C}^{n\times n} \ : \ A + A^* \in \overline{\mathbf{P}}_n \right\} \qquad \overline{\mathbf{S}}_{I_n} := \{ A \in \mathbb{C}^{n\times n} \ : \ 1 \geq \|A\|_2 \}. \tag{2.5}$$

There is structural gap between $\mathbf{L}_H$ with $H \in \mathbf{H}_n$, and the special case $\mathbf{L}_{I_n}$. This was addressed in [25, Theorems 2.4, 4.1]. Similarly the structural gap between $\mathbf{S}_H$, and the special case $\mathbf{S}_{I_n}$ was addressed in [26, Theorem 2.1, Proposition 3.4].

We find it convenient to employ this matrix notation in order to extend to matrix-valued set-up, the description of the above four families of scalar real rational functions

Definition 2.1. Consider the following four families of $m \times m$-valued real rational function.

- (α) $F \in \mathcal{P}$, means that $\forall z \in \mathbb{C}_R$ one has that $F(z) \in \overline{\mathbf{L}}_{I_m}$.
- (β) $F \in \mathcal{B}$, means that $\forall z \in \mathbb{C}_R$ one has that $F(z) \in \overline{\mathbf{S}}_{I_m}$.
- (γ) $F \in \mathcal{DP}$ means that $\forall z \in \mathbb{D}$ one has that $F(z) \in \overline{\mathbf{L}}_{I_m}$.
- (δ) $F \in \mathcal{DB}$ means that $\forall z \in \mathbb{D}$ one has that $F(z) \in \overline{\mathbf{S}}_{I_m}$.

□

For completeness we next combine Definition 2.1 along with the Cayley transform. To this end recall that the Cayley transform of $m \times m$-valued rational function $F(z)$, is defined as,

$$\mathcal{C}(F) = (I_m - F)(I_m + F)^{-1} \qquad \det\left(F(z) + I_m\right) \not\equiv 0 \quad \forall z \in \mathbb{C}.$$

The functions in Definition 2.1 satisfy the following relations,

$$\begin{array}{lccc} & \mathcal{B} = \mathcal{C}\left(\mathcal{P}\right) & & \mathcal{DB} = \mathcal{C}\left(\mathcal{DP}\right) \\ (2.6) & F(z) \in \mathcal{P} & \Longleftrightarrow & F\left(\frac{1+z}{1-z}\right) \in \mathcal{DP} \\ & F(z) \in \mathcal{B} & \Longleftrightarrow & F\left(\frac{1+z}{1-z}\right) \in \mathcal{DB}. \end{array}$$

(In the Mathematical analysis terminology the formulation is more symmetric, e.g. $F(z)$ belongs to $\mathcal{P}$ is equivalent to having $F\left(\mathcal{C}(z)\right)$ a Herglotz. function).

3. Matrix-convex sets

We next resort to the notion of a *matrix-convex* set, see e.g. [18] and more recently, [19], [20], [31].

Definition 3.1. A family $\mathbf{A}$, of square matrices (of various dimensions) is said to be *matrix-convex of level* n, if for all $\nu = 1, \ \ldots \ , \ n$:
For all natural k,

$$\sum_{j=1}^{k} \Upsilon_j^* \Upsilon_j = I_\nu \qquad \begin{smallmatrix} \forall \Upsilon_j \in \mathbb{C}^{\eta_j \times \nu} \\ \eta_j \in [1, \ \nu], \end{smallmatrix}$$

having $A_1, \ \ldots \ , \ A_k$ (of various dimensions 1×1 through $\nu \times \nu$) within $\mathbf{A}$, implies that also

$$\sum_{j=1}^{k} \Upsilon_j^* A_j \Upsilon_j \ ,$$

belongs to $\mathbf{A}$.

If the above holds for all n, we say that the set $\mathbf{A}$ is *matrix-convex.* □

In the rest of the section we briefly explore the notion of matrix-convexity.

Lemma 3.2. [25]. *The following sets are matrix-convex:*

- (i) $\overline{\mathbf{H}}$, $\quad i\overline{\mathbf{H}}$, $\quad \overline{\mathbf{P}}$, $\quad \mathbf{P}$
- (ii) $\{ \ A \ : \ \text{Bound} \geq \|A\|_2 \}$ *for some* Bound > 0.
- (iii) *The (open) closed set* ($\mathbf{L}_I$) $\overline{\mathbf{L}}_I$, *see Eq.* (2.5).

Recall that the matrix-convexity condition is quite restrictive, so there are not-too-many, non-trivial sets with this property. For example, the sets (i) Toeplitz matrices, (ii) $\{ \ A \ : \ \text{Bound} \geq \|A\|_1 \}$ for some Bound > 0, (iii) $\mathbf{L}_P$ with $\alpha I \neq P \in \mathbf{P}$, are convex, but not matrix-convex. Furthermore, matrix-convexity implies both classical convexity and being unitarily-invariant, but the combination of these two properties still falls short of characterizing matrix-convexity. Indeed the set of positively scalar matrices, i.e of the form αI, $\alpha > 0$ is unitarily invariant and convex. However, it is not matrix-convex: $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ belong to the set but not the following combination where

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\Upsilon_1^*}\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{A_1}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\Upsilon_1}+\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Upsilon_2^*}\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}}_{A_2}\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Upsilon_2}=\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad {\Upsilon_1}^*\Upsilon_1+{\Upsilon_2}^*\Upsilon_2=I_2 \ .$$

Nevertheless, the four families of passive rational functions we focus on, do share this property.

Proposition 3.3. *Each of the rational functions sets $\mathcal{P}$, $\mathcal{B}$, $\mathcal{DP}$ and $\mathcal{DB}$ is matrix-convex.*

Proof : Let $F(z)$ be in $\mathcal{P}$ or in $\mathcal{B}$ and let $F(z_o)$ be the image of a point z_o which lies in the domain of interest ($\mathbb{C}_R$ and $\overline{\mathbb{D}}^c$ for $\mathcal{P}$ and $\mathcal{B}$, respectively) From items (α), (γ) in Definition 2.1 it follows that as a matrix, $F(z_o)$ is in $\overline{\mathbf{L}}_I$, see Eq. (2.5), which is matrix-convex by item (iii) of Lemma 3.2.

In a similar way, let $F(z)$ be in $\mathcal{DP}$ or in $\mathcal{DB}$, and let $F(z_o)$ be the image of a point z_o which lies in the domain of interest ($\mathbb{C}_R$ and $\overline{\mathbb{D}}^c$ for $\mathcal{DP}$ and $\mathcal{DB}$, respectively). From items (β), (δ) in Definition 2.1 it follows that as a matrix, $F(z_o)$ is in $\overline{\mathbf{S}}_I$, see Eq. (2.5), which is matrix-convex by item (ii) of Lemma 3.2. □

In Proposition 6.3 below, we offer a statement analogous to Proposition 3.3, but in the framework of realization arrays.

We end this section by pointing out that one can go beyond Proposition 3.3.

Theorem 3.4. *(I)* [25]. *The family $\mathcal{P}$, of $m \times m$-valued positive real rational functions, is a cone, closed under inversion and a maximal matrix-convex family of functions which is analytic in $\mathbb{C}_R$.*

Conversely, a maximal matrix-convex cone of $m \times m$-valued rational functions, analytic in $\mathbb{C}_R$, containing the zero degree function $F(z) \equiv I_m$, is the set $\mathcal{P}$.

(II) [26]. *A family of $m \times m$-valued real rational functions $F(z)$ which for all $z \in \overline{\mathbb{D}}^c$ is: Analytic, matrix-convex and a maximal set closed under multiplication among its elements, is the set $\mathcal{DB}$.*

The converse is true as well.

4. Characterization through State-Space: A Unified Framework

We start by recalling in the classical Kalman-Yakubovich-Popov type characterization, through state-space realization, of the families in Definition 2.1.

Theorem 4.1. *Consider the four families of $m \times m$-valued rational functions in Definition 2.1. For each $l = \alpha$, β, γ and δ, let R_{F_l} be a corresponding state-space realization,*

$$F_l(z) = C_l(zI_n - A_l)^{-1}B_l + D_l \ .$$

(I) Assume Tet there exist $n \times n$ positive definite matrices P so that the resulting $(n+m) \times (n+m)$ matrices Q_l are positive semi-definite.

(α) *For*

$$Q_\alpha := \begin{pmatrix} -PA-A^*P & C^*-PB \\ C-B^*P & D+D^* \end{pmatrix},$$

the function $F_\alpha(z)$ *is Positive-Real. See e.g.* [3], [6, Chapter 5], [11, Chapter 7], [12, Subsection 2.7.2], [17], *and* [37, Theorem 3].

(β) *For*

$$Q_\beta := \begin{pmatrix} -PA-A^*P & -PB \\ -B^*P & I_m \end{pmatrix} - \begin{pmatrix} C^* \\ D^* \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix},$$

the function $F_\beta(z)$ *is Bounded-Real. See e.g.* [5], [6, Section 7.2], [11, Chapter 7], [12, Subsection 2.7.3], *and* [17]

(γ) *For*

$$Q_\gamma := \begin{pmatrix} P & C^* \\ C & D+D^* \end{pmatrix} - \begin{pmatrix} A^* \\ B^* \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix},$$

the function $F_\gamma(z)$ *is Discrete-Time-Positive-Real. See e.g.* [21], [24], *and* [39, Lemma 1].

(δ) *For,*

$$Q_\delta ::= \begin{pmatrix} P-A^*PA & -A^*PB \\ -B^*PA & I_m \end{pmatrix} - \begin{pmatrix} C^* \\ D^* \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix},$$

the function $F_\delta(z)$ *is Discrete-Time-Bounded-Real. See e.g.* [30], [32], [36] *and* [39, Lemma 7].

(II) In each of the four above cases, $l = \alpha, \beta, \gamma, \delta$, *if the state-space realization* $F_l(z) = C_l(zI_n - A_l)^{-1}B_l + D_l$, *is minimal, then the converse is true as well.*

Remark 4.2. For completeness we recall in three extension of Theorem 4.1, which are beyond the scope of this work.

1. If in each Q_l in Theorem 4.1, having $P \in \mathbf{P}_n$ is relaxed to $H \in \mathbf{H}_n$, then Generalized-positivity (boundedness, ...) is obtained. For more details see [3], [5],[10, Theorem 10.2] and [17].

2. If each Q_l in Theorem 4.1 is restricted to belong to $\mathbf{P}_{n+m}$, then the stronger notion of Hyper-positivity (Hyper-boundednes, ...) is obtained. For further details see [4].

3. In [4] we addressed quantitative subsets of $\mathcal{B}$, i.e. functions where,

$$\sqrt{\tfrac{\eta-1}{\eta+1}} \geq \sup_{z\in\mathbb{C}_R} \|F(z)\|_2 \qquad \eta \in (1,\ \infty].$$

A state-space characterization of the above F belongs to this family when Q_β is substituted by $Q_\beta(\eta) = \begin{pmatrix} -PA-A^*P & -PB \\ -B^*P & I_m \end{pmatrix} - \frac{\eta+1}{\eta-1} \begin{pmatrix} C^* \\ D^* \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix}$, which is positive-semidefinite.

Note that $\mathcal{B}$ is recovered when $\eta \rightarrow \infty$. □

To establish a unified framework, let us construct four $2(n+m) \times 2(n+m)$ matrices, starting from

$$W_\delta = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}. \tag{4.1}$$

Next taking a real symmetric orthogonal matrix $U = \begin{pmatrix} \frac{1}{\sqrt{2}}I_n & 0 & \frac{1}{\sqrt{2}}I_n & 0 \\ 0 & I_m & 0 & 0 \\ \frac{1}{\sqrt{2}}I_n & 0 & -\frac{1}{\sqrt{2}}I_n & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$, yields

$$U^*W_\delta U = W_\beta = \begin{pmatrix} 0 & 0 & -P & 0 \\ 0 & -I_m & 0 & 0 \\ -P & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}. \tag{4.2}$$

Now taking the permutation matrix $U = \begin{pmatrix} 0 & -I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}$, one can obtain,

$$W_\delta U = W_\alpha = \begin{pmatrix} 0 & 0 & -P & 0 \\ 0 & 0 & 0 & I_m \\ -P & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix} \tag{4.3}$$

and

$$W_\beta U = W_\gamma = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & P & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}. \tag{4.4}$$

Remark 4.3. By construction the three real symmetric matrices: W_α, W_β and W_γ are orthogonally similar to the real, block-diagonal, symmetric matrix W_δ. In particular the spectrum of each W matrix, is that of $\pm P$ together with $\pm I_m$. □

The verification of the following, amounts to straightforward computation.

Lemma 4.4. *Let W_α, W_β, W_γ and W_δ be as in Eqs.* (4.3), (4.2), (4.4) *and* (4.1), *respectively then,*

$$Q_l = \begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix}^* W_l \begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix} \qquad l = \alpha, \beta, \gamma, \delta, \tag{4.5}$$

with Q_l as in Theorem 4.1.

This Lemma enables us now to cast the four items of Theorem 4.1 in a uniform framework.

Theorem 4.5. *Let R_{F_l} be an $(n+m) \times (n+m)$ realization of $m \times m$-valued rational function, $F_l(z)$,*

$$F_l(z) = C_l(zI_n - A_l)^{-1}B_l + D_l \qquad R_{F_l} = \left(\begin{array}{c|c} A_l & B_l \\ \hline C_l & D_l \end{array}\right) \qquad l = \alpha, \beta, \gamma, \delta.$$

(I) With W_α, W_β, W_γ and W_δ from Eqs. (4.3), (4.2), (4.4) *and* (4.1), *respectively consider the relation,*

$$\underbrace{\begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix}^* W_l \begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix}}_{Q_l \text{ in Eq. (4.5)}} \in \overline{\mathbf{P}}_{n+m} . \tag{4.6}$$

Then the following is true:

- (α) *If the condition in Eq.* (4.6) *is satisfied for W_α then $F_\alpha(z)$ is a Positive-Real function.*
- (β) *If the condition in Eq.* (4.6) *is satisfied for W_β then $F_\beta(z)$ is a Bounded-Real function.*
- (γ) *If the condition in Eq.* (4.6) *is satisfied for W_γ then $F_\gamma(z)$ is a Discrete-Time-Positive-Real function.*
- (δ) *If the condition in Eq.* (4.6) *is satisfied for W_δ then $F_\delta(z)$ is a Discrete-Time-Bounded-Real function.*

(II) In each of the four above cases, if the realization R_F is minimal, then the converse is true as well.

We next illustrate an application of the unified framework in Theorem 4.5.

Example 4.6. We here address two forms of inversion and then examine the behavior of the families $\mathcal{P}$ and $\mathcal{DB}$ with respect to these two inversions.

We start with the classical one: When $F(z)$ is $m \times m$-valued rational function, so that $\det(F(z)) \not\equiv 0$, one can define $(F(z))^{-1}$ so that $(F(z))^{-1} F(z) = I_m$ almost everywhere in $\mathbb{C}$.

Now, if $F \in \mathcal{P}$, by Theorem 3.4(I) F^{-1} belongs to $\mathcal{P}$ as well. In contrast, if $F \in \mathcal{DB}$, from Theorem 3.4(II) it follows that F^{-1} is in fact an "anti"-$\mathcal{DB}$ function in the sense that $\inf\limits_{z \in \overline{\mathbb{D}}^c} \sigma_m\left((F(z))^{-1}\right) > 1$, where $\sigma_m(X)$ is the smallest singular value of a matrix $X \in \mathbb{C}^{m \times m}$.

As a simple illustration take $f(z) = \frac{1}{2(2z+1)}$ (small letters for scalar functions) and $(f(z))^{-1} = 2(2z+1)$. Then, on the one hand both f and f^{-1} are in $\mathcal{P}$. Now the same f is in $\mathcal{DB}$ but f^{-1} belongs to "anti"-$\mathcal{DB}$ in the sense that $|\,(f(z))^{-1}\,| > 1$ for all $|z| > 1$.

One can now define another form of inversion, yielding an $m \times m$-valued rational function (typically) of McMillan degree n: Let $F(z)$ and R_F be as in Theorem 4.5. Treating the realization array R_F as an $(n+m) \times (n+m)$ matrix, whenever it is non-singular. Then $R_G = R_F^{-1}$ so that $R_G R_F = I_{n+m}$.

Let us return to the above simple illustration with $f(z) = \frac{1}{2(2z+1)}$ so that its minimal realization is $R_f = \frac{1}{2}\left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right)$ and then $R_g = R_f^{-1} = 2\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right)$, which in turn means that $g(z) = \frac{2(2+z)}{z}$.

Now both $f(z) = \frac{1}{2(2z+1)}$ and $g(z) = \frac{2(2+z)}{z}$ are positive real. However, while $f(z)$ is also a $\mathcal{DB}$ function, $g(z)$ can not be a $\mathcal{DB}$ function (nor "anti"-$\mathcal{DB}$) since it maps $\{\mathbb{R} \smallsetminus [-1,\ 1]\}$ to $(-2, 6)$.

Next, in the general framework, multiplying Eqs. (4.5), (4.6) by R_G (recall $= (R_F)^{-1}$) from the left and by $R_G{}^*$ from the right, yields

$$
\begin{aligned}
\underbrace{R_G{}^* Q_l R_G}_{\tilde{Q}_l} &= R_G{}^* \begin{pmatrix} R_F \\ I_{n+m} \end{pmatrix}^* W_l \begin{pmatrix} R_F \\ I_{n+m} \end{pmatrix} R_G = \begin{pmatrix} I_{n+m} \\ R_G \end{pmatrix}^* W_l \begin{pmatrix} I_{n+m} \\ R_G \end{pmatrix} \\
&= \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}}_{U} W_l \underbrace{\begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}}_{U} \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}.
\end{aligned}
\tag{4.7}
$$

From Eq. (4.3) it follows that with $U = \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}$ one has that $U^* W_\alpha U = W_\alpha$ while $U^* W_\delta U = -W_\delta$. In items (a) and (b) below, we examine the system interpretation of this technical observation.

(a) If in Eq. (4.7) $F(z)$ is positive real, i.e. $l = \alpha$, one has that

$$
\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} W_\alpha \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}}_{U^* W_\alpha U = W_\alpha} \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} = \tilde{Q}_\alpha \in \overline{\mathbf{P}}_{n+m} \ ,
$$

namely, $\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* W_\alpha \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} \in \overline{\mathbf{P}}_{n+m}$. One can conclude that also $G(z)$ is positive real.

(b) If in Eq. (4.7) $F \in \mathcal{DB}$, i.e. $l = \delta$, one has that

$$\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} W_\delta \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}}_{U^* W_\delta U = -W_\delta} \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} = \tilde{Q}_\delta ,$$

namely, $\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* (-W_\delta) \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} \in \overline{\mathbf{P}}_{n+m}$. Thus, one can say that $G(z)$ is "anti"-$\mathcal{DB}$: More precisely, $\inf\limits_{z \in \mathbb{D}^c} \sigma_m \left((F(z))^{-1} \right) > 1$. □

Consider the four families $\mathcal{P}$, $\mathcal{B}$, $\mathcal{DP}$ and $\mathcal{DB}$. As already mentioned, as *rational functions* they are related through the Cayley transform, see Eq. (2.6). Theorem 4.5 suggests an additional inter-relations: Through the corresponding *state-space realizations*. This is pursued in Section 6

5. Two extreme cases

In this section we review two disjoint *subsets* of each of the four families $\mathcal{P}$, $\mathcal{B}$, $\mathcal{DP}$ and $\mathcal{DB}$, which satisfy a Kalman-Yakubovich-Popov type characterization in the form of Theorem 4.5.

To ease the reading, we present only samples of the four cases.

5.1. **Lossless.** Recall that the subset of $\mathcal{P}$ rational functions $F(z)$, which in addition satisfy,

$$\left((F(z))^* + F(z) \right)_{|z \in i\mathbb{R}} = 0,$$

is called *Lossless Positive* (a.k.a. Positive Real Odd or Foster), denoted by $\mathcal{LP}$.

In electrical circuits framework, $\mathcal{LP}$ functions are the driving point impedance of $L - C$ network, i.e. a circuit comprised of inductors and capacitors only (with zero resistance), see e.g. [6, Theorem 2.7.4], [11, Ch 8, item 36], [15, Section 4.2] and [25].

Following the relations in Eq. (2.6) in a similar way, one may focus on the subset of $m \times m$-valued $\mathcal{B}$ rational functions $F(z)$, which in addition satisfy,

$$(F(z))^* F(z)_{|z \in i\mathbb{R}} = I_m ,$$

and call it Lossless Bounded, denoted by $\mathcal{LB}$.

Now, items (α) and (β) of Theorem 4.5 takes the following form.

Proposition 5.1. *Let R_F be an $(n+m) \times (n+m)$ realization of $m \times m$-valued rational function, $F(z)$,*

$$F(z) = C(zI_n - A)^{-1}B + D \qquad R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

If,

$$\begin{pmatrix} R_F \\ I_{n+m} \end{pmatrix}^* W_\alpha \begin{pmatrix} R_F \\ I_{n+m} \end{pmatrix} = 0_{n \times n} ,$$

with W_α as in Eq. (4.3), then $F(z)$ is a $\mathcal{LP}$ function.

If,

$$\begin{pmatrix} R_F \\ \\ I_{n+m} \end{pmatrix}^* W_\beta \begin{pmatrix} R_F \\ \\ I_{n+m} \end{pmatrix} = 0_{n\times n} \ ,$$

with W_β as in Eq. (4.2)*, then $F(z)$ is a $\mathcal{LB}$ function.*

For further details see e.g. [1], [2] and [6, p. 221-222, p.312].

5.2. **Hyper-Bounded.** In [4] we studied Hyper Bounded real rational:
A $m \times m$-valued rational function is said to be η-Hyper Bounded, denoted by $\mathcal{HB}_\eta$, for some $\eta \in (1, \ \infty]$ if,

$$\sqrt{\tfrac{\eta-1}{\eta+1}} \geq \sup_{z\in\mathbb{C}_R} \|F(z)\|_2 \ .$$

Equivalently (in the framework of Definition 2.1), $F \in \mathcal{HB}_\eta$ if

$$F(z) \in \sqrt{\tfrac{\eta-1}{\eta+1}} \cdot \overline{\mathbf{S}}_{I_m} \qquad \forall z \in \mathbb{C}_R \ .$$

This definition introduces the following partial ordering,

$$\infty > \eta > \hat{\eta} > 1 \qquad \Longrightarrow \qquad \mathcal{HB}_{\hat{\eta}} \subset \mathcal{HB}_\eta \subset \{\mathcal{B} \smallsetminus \mathcal{LB}\}.$$

Here, W_β in Eq. (4.2) is refined to,

$$W_{\beta,\eta} = \begin{pmatrix} 0 & 0 & -P & 0 \\ 0 & \frac{1+\eta}{1-\eta}I_m & 0 & 0 \\ -P & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix} \qquad \eta \in (1, \ \infty]. \tag{5.1}$$

Namely, $W_\beta = \lim\limits_{\eta \to \infty} W_{\beta,\eta}$.

Now, item (β) of Theorem 4.5 takes the following form.

Proposition 5.2. *Let R_F be an $(n+m) \times (n+m)$ realization of $m \times m$-valued rational function, $F(z)$,*

$$F(z) = C(zI_n - A)^{-1}B + D \qquad R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

If,

$$\begin{pmatrix} R_F \\ \\ I_{n+m} \end{pmatrix}^* W_{\beta,\eta} \begin{pmatrix} R_F \\ \\ I_{n+m} \end{pmatrix} \in \overline{\mathbf{P}}_{n+m} \ ,$$

with $W_{\beta,\eta}$ as in Eq. (5.1)*, then $F(z)$ is a $\mathcal{HB}_\eta$ function.*

For details, see Lemma 2.3 and Eq. (2.6) in [4].

For completeness, we point out that as before, one can definite analogue sets by mimicking the relations in Eq. (2.6). For example, a $m \times m$-valued rational function is said to be η-Hyper Discrete Bounded, denoted by $\mathcal{HDB}_\eta$, for some $\eta \in (1, \ \infty]$ if

$$\sqrt{\tfrac{\eta-1}{\eta+1}} \geq \sup_{z\in\overline{\mathbb{D}}^c} \|F(z)\|_2 \ .$$

We shall not further pursue this direction.

6. Sets of Matrix-convex Realization arrays

In this section we address inter-relations within *families of realization arrays* associated with rational functions. As a preliminary step we recall in the classical notion of transformation of coordinates: Substituting a given state-space realization R_F by $\left(\begin{smallmatrix} T^{-1} & 0 \\ 0 & I_m \end{smallmatrix}\right) R_F \left(\begin{smallmatrix} T & 0 \\ 0 & I_m \end{smallmatrix}\right)$, for some non-singular $n \times n$ matrix T.

Lemma 6.1. *Consider the framework of Theorem 4.5 for some* $l \in \{\alpha, \beta, \gamma, \delta\}$.
Up to a change of coordinates, one can take in Eq. (4.5),

$$\begin{pmatrix} R_{F_l} \\ \\ I_{n+m} \end{pmatrix}^* \hat{W}_l \begin{pmatrix} R_{F_l} \\ \\ I_{n+m} \end{pmatrix} \in \overline{\mathbf{P}}_{n+m} \ , \tag{6.1}$$

where the $\hat{W}$*'s are associated with balanced realization, i.e.*

$$\begin{array}{ll} \hat{W}_\alpha = \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \\ -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix} & \hat{W}_\beta = \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & -I_m & 0 & 0 \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix} \\ \hat{W}_\gamma = \begin{pmatrix} -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix} & \hat{W}_\delta = \begin{pmatrix} -I_n & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}. \end{array} \tag{6.2}$$

A system whose realization satisfies Eq. (6.1) with $\hat{W}_\alpha$ from Eq. (6.2) is called "internally passive", see [38, Definition 3]

We also need to resort to following.

Definition 6.2. For all k, let $v_j \in \mathbb{C}^{(n+m)\times(n+m)}$, $j = 1, \ldots, k$ be block-diagonal so that

$$\sum_{j=1}^{k} \underbrace{\begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix}^*}_{\Upsilon_j^*} \underbrace{\begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix}}_{\Upsilon_j} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}. \tag{6.3}$$

A set $\mathbf{R}$ of $(n+m)\times(n+m)$ matrices is said to be *n, m-matrix-convex* if having $R_1, \ldots, R_k$ in $\mathbf{R}$, implies that also

$$R_F := \sum_{j=1}^{k} \underbrace{\begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix}^*}_{\Upsilon_j^*} \underbrace{\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}}_{R_{F_j}} \underbrace{\begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix}}_{\Upsilon_j}, \tag{6.4}$$

belongs to $\mathbf{R}$ for all natural k and all $\Upsilon_j \in \mathbb{C}^{(n+m)\times(n+m)}$. □

In [25] it was pointed out that the notion of *n, m-matrix-convexity* is intermediate between (the more strict) *matrix-convexity*, and (weaker) classical convexity.

We now pose the following question: For a natural parameter k, let $F_1(s), \ldots, F_k(s)$ be a family of $m \times m$-valued rational functions all from the same family, admitting $(n+m)\times(n+m)$ realizations, i.e.

$$R_{F_j} = \left(\begin{array}{c|c} \hat{A}_j & \hat{B}_j \\ \hline \hat{C}_j & \hat{D}_j \end{array}\right) \qquad j = 1, \ldots, k. \tag{6.5}$$

Let R_F, a realization of an $m \times m$-valued rational function $F(z)$, be as in Eq. (6.4). We now address the following problem:

Under what conditions having $F_1(z), \ \dots \ , \ F_k(z)$ in Eq. (6.5) all in $\mathcal{P}$ (or $\mathcal{B}$ or $\mathcal{DP}$ or $\mathcal{DB}$) implies that also the resulting $F(z)$ in Eq. (6.4) belongs to the same set?

If such a property holds this suggests that out of a small number of "extreme points" of balanced realizations of $\mathcal{P}$ (or $\mathcal{B}$ or $\mathcal{DP}$ or $\mathcal{DB}$) rational functions, one can construct a whole "matrix-convex-hull" realizations of functions within the same family. As a sample application, this may enable one to perform a simultaneous balanced truncation model order reduction of a whole family of bounded real functions, in the spirit of [14, Section 5].

Before addressing this question, a word of caution: For example, $R_1 = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $R_2 = \left(\begin{array}{c|c} A & -B \\ \hline -C & D \end{array}\right)$ are two realization of the same rational function. Furthermore, R_1 is minimal (balanced) if and only if R_2 is minimal (balanced). However, $R_3 = \frac{1}{2}(R_1 + R_2) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$ is only a non-minimal realization of a zero degree rational function $F(s) \equiv D$.

In a similar way, even when the "extreme points" realizations in Eq. (6.5) are all balanced, the resulting realization R_F in Eq. (6.4), may be not minimal.

We now return to the above question,

Proposition 6.3. *Consider the framework of Lemma 6.1 where* $l \in \{\alpha, \ \beta, \gamma, \delta\}$ *is prescribed. For a natural parameter* k*, let* $F_{1,l}(z) \ , \ \dots \ , \ F_{k,l}(z)$ *be a family of* $m \times m$*-valued rational functions, admitting* $(n+m) \times (n+m)$ *realizations as in Eq.* (6.5)*, satisfying all Eq.* (6.1) *i.e.*

$$\begin{pmatrix} R_{F_{j,l}} \\ \\ I_{n+m} \end{pmatrix}^* \hat{W}_l \begin{pmatrix} R_{F_{j,l}} \\ \\ I_{n+m} \end{pmatrix} = Q_{j,l} \in \overline{\mathbf{P}}_{n+m} \qquad \begin{matrix} l \in \{\alpha, \ \beta, \ \gamma, \ \delta\} \quad \text{prescribed} \\ \\ j = 1, \ \dots \ , \ k. \end{matrix} \tag{6.6}$$

Then, R_F *in Eq. 6.4 satisfies the same relation, i.e. each of sets* $\mathcal{P}$*,* $\mathcal{B}$*,* $\mathcal{DP}$ *and* $\mathcal{DB}$ *is a realization-*m, n*-matrix-convex.*

Proof : Assume that Eq. (6.6) holds for $l = \alpha$, i.e.

$$\underbrace{\begin{pmatrix} A_j & B_j \\ C_j & D_j \\ I_n & 0 \\ 0 & I_m \end{pmatrix}^* \overbrace{\begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \\ -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}}^{\hat{W}_\alpha} \begin{pmatrix} A_j & B_j \\ C_j & D_j \\ I_n & 0 \\ 0 & I_m \end{pmatrix}}_{Q_j} \qquad \begin{matrix} j{=}1, \ \dots \ , \ k \\ Q_j \in \overline{\mathbf{P}}_{n+m}, \end{matrix}$$

and consider matrix-convex combination of realizations as in Eq. (6.4),

$$\begin{pmatrix} \sum\limits_{j=1}^{k} \Upsilon_{j,n}^* A_j \Upsilon_{j,n} & \sum\limits_{j=1}^{k} \Upsilon_{j,n}^* B_j \Upsilon_{j,m} \\ \sum\limits_{j=1}^{k} \Upsilon_{j,m}^* C_j \Upsilon_{j,n} & \sum\limits_{j=1}^{k} \Upsilon_{j,m}^* D_j \Upsilon_{j,m} \\ I_n & 0 \\ 0 & I_m \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \\ -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}}_{\hat{W}_\alpha} \begin{pmatrix} \sum\limits_{j=1}^{k} \Upsilon_{j,n}^* A_j \Upsilon_{j,n} & \sum\limits_{j=1}^{k} \Upsilon_{j,n}^* B_j \Upsilon_{j,m} \\ \sum\limits_{j=1}^{k} \Upsilon_{j,m}^* C_j \Upsilon_{j,n} & \sum\limits_{j=1}^{k} \Upsilon_{j,m}^* D_j \Upsilon_{j,m} \\ I_n & 0 \\ 0 & I_m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{k} \Upsilon_{j,n}^* A_j \Upsilon_{j,n} & \sum_{j=1}^{k} \Upsilon_{j,n}^* B_j \Upsilon_{j,m} \\ \sum_{j=1}^{k} \Upsilon_{j,m}^* C_j \Upsilon_{j,n} & \sum_{j=1}^{k} \Upsilon_{j,m}^* D_j \Upsilon_{j,m} \\ I_n & 0 \\ 0 & I_m \end{pmatrix}^* \begin{pmatrix} 0 & 0 & -\sum_{j=1}^{k} \Upsilon_{j,n}^* \Upsilon_{j,n} & 0 \\ 0 & 0 & 0 & \sum_{j=1}^{k} \Upsilon_{j,m}^* \Upsilon_{j,m} \\ -\sum_{j=1}^{k} \Upsilon_{j,n}^* \Upsilon_{j,n} & 0 & 0 & 0 \\ 0 & \sum_{j=1}^{k} \Upsilon_{j,m}^* \Upsilon_{j,m} & 0 & 0 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{k} \Upsilon_{j,n}^* A_j \Upsilon_{j,n} & \sum_{j=1}^{k} \Upsilon_{j,n}^* B_j \Upsilon_{j,m} \\ \sum_{j=1}^{k} \Upsilon_{j,m}^* C_j \Upsilon_{j,n} & \sum_{j=1}^{k} \Upsilon_{j,m}^* D_j \Upsilon_{j,m} \\ I_n & 0 \\ 0 & I_m \end{pmatrix}$$

$$= \sum_{j=1}^{k} \begin{pmatrix} \Upsilon_{j,n}^* & 0 \\ 0 & \Upsilon_{j,m}^* \end{pmatrix} \underbrace{\begin{pmatrix} A_j & B_j \\ C_j & D_j \\ I_n & 0 \\ 0 & I_m \end{pmatrix}^* \overbrace{\begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \\ -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}}^{\hat{W}_\alpha} \begin{pmatrix} A_j & B_j \\ C_j & D_j \\ I_n & 0 \\ 0 & I_m \end{pmatrix}}_{Q_j} \begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix}$$

$$= \sum_{j=1}^{k} \begin{pmatrix} \Upsilon_{j,n}^* & 0 \\ 0 & \Upsilon_{j,m}^* \end{pmatrix} Q_j \begin{pmatrix} \Upsilon_{j,n} & 0 \\ 0 & \Upsilon_{j,m} \end{pmatrix} \in \overline{\mathbf{P}}_{n+m} \,.$$

Thus the case of $l = \alpha$ is established.

Since $\hat{W}_\beta$, $\hat{W}_\gamma$ and $\hat{W}_\delta$ are just permutations of $\hat{W}_\alpha$, the respective constructions are very similar and thus omitted, and the proof is complete. □

Special cases of Proposition 6.3 for $\mathcal{P}$ and for $\mathcal{DB}$ were shown in [25] and [26], respectively.

We next illustrate how, by using the above results, one can generate from a single system, a whole collection of them. For simplicity, we address

Example 6.4. Consider the three following 2×2-valued $\mathcal{LP}$ rational functions along with the corresponding balanced realizations, where $a, b \in \mathbb{R}$ are parameters.

$$F_1(z) = \tfrac{1}{a^2 z} \begin{pmatrix} \frac{b^2}{a^2} + 1 & z - \frac{b}{a} \\ -(z + \frac{b}{a}) & 1 \end{pmatrix} \qquad R_{F_1} = \left(\begin{array}{cc|cc} 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{b}{a^2} & -\frac{1}{a} \\ \hline \frac{1}{a} & \frac{b}{a^2} & 0 & \frac{1}{a^2} \\ 0 & -\frac{1}{a} & -\frac{1}{a^2} & 0 \end{array}\right)$$

$$(6.7) \qquad F_2(z) = \tfrac{1}{z^2+1} \begin{pmatrix} a^2 z & a(bz - a) \\ a(bz + a) & (a^2 + b^2) z \end{pmatrix} \qquad R_{F_2} = \left(\begin{array}{cc|cc} 0 & 1 & a & b \\ -1 & 0 & 0 & -a \\ \hline a & 0 & 0 & 0 \\ b & -a & 0 & 0 \end{array}\right)$$

$$F_3(z) = \tfrac{z}{1+z^2} \begin{pmatrix} 1 & \frac{b}{a} - z \\ \frac{b}{a} + z & \frac{b^2}{a^2} + 1 \end{pmatrix} \qquad R_{F_3} = \left(\begin{array}{cc|cc} 0 & 1 & 1 & \frac{b}{a} \\ -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 \\ \frac{b}{a} & 1 & 1 & 0 \end{array}\right).$$

Following the first part of Section 5, each of these three functions satisfy,

$$\begin{matrix} -F(z) \in \overline{\mathbf{L}}_{I_2} & z \in \mathbb{C}_L \\ F(z) \in i\overline{\mathbf{H}}_2 & z \in i\mathbb{R} \\ F(z) \in \overline{\mathbf{L}}_{I_2} & z \in \mathbb{C}_R \end{matrix} \qquad \text{and/or} \qquad \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} R_F + {R_F}^* \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = 0_{4\times 4} \,.$$

To employ Proposition 3.3 to generate additional rational functions, let now $\Upsilon_j \in \mathbb{C}^{2\times 2}$ be arbitrary so that $\sum_{j=1}^{3} \Upsilon_j^* \Upsilon_j = I_2$. Then, with $F_j(s)$ from Eq. (6.7), one has that $\sum_{j=1}^{3} \Upsilon_j^* F_j(z) \Upsilon_j$ is a 2×2-valued positive real odd rational function.

Similarly, to generate additional systems by employing Proposition 6.3 let now $\tilde{\Upsilon}_j, \tilde{\tilde{\Upsilon}}_j \in \mathbb{C}^{2\times 2}$ be arbitrary so that $\sum_{j=1}^{3} \tilde{\Upsilon}_j^* \tilde{\Upsilon}_j = I_2$ and $\sum_{j=1}^{3} \tilde{\tilde{\Upsilon}}_j^* \tilde{\tilde{\Upsilon}}_j = I_2$. Then, with R_{F_j} from Eq. (6.7), one has that $\sum_{j=1}^{3} \begin{pmatrix} \tilde{\Upsilon}_j & 0 \\ 0 & \tilde{\tilde{\Upsilon}}_j \end{pmatrix}^* R_{F_j} \begin{pmatrix} \tilde{\Upsilon}_j & 0 \\ 0 & \tilde{\tilde{\Upsilon}}_j \end{pmatrix}$ is a $(2+2) \times (2+2)$ realization (recall, not necessarily minimal) of a positive real odd rational function.

Finally note that we actually started from a single system F_1. Indeed, F_2 is defined as, $R_{F_2} = (R_{F_1})^{-1}$ (in the sense of inverting a constant 4×4 matrix). Now, $F_3(z) = (F_1(z))^{-1}$ (in the sense of that the product of pair of rational functions, each of degree two, yields a zero degree rational function, i.e. $F_3(z)F_1(z) \equiv I_2$). □

We conclude by remarking that, following Section 5 and Example 6.4, one can formally specialize Proposition 6.3 to balanced realizations of $\mathcal{LB}$ and of $\mathcal{HB}_\eta$ functions.

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