

A NOTE ON BANACH SPACES E ADMITTING A CONTINUOUS MAP FROM $C_p(X)$ ONTO E_w

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ABSTRACT. $C_p(X)$ denotes the space of continuous real-valued functions on a Tychonoff space X endowed with the topology of pointwise convergence. A Banach space E equipped with the weak topology is denoted by E_w . It is unknown whether $C_p(K)$ and $C(L)_w$ can be homeomorphic for infinite compact spaces K and L [14], [15]. In this paper we deal with a more general question: what are the Banach spaces E which admit certain continuous surjective mappings $T : C_p(X) \rightarrow E_w$ for an infinite Tychonoff space X ?

First, we prove that if T is linear and sequentially continuous, then the Banach space E must be finite-dimensional, thereby resolving an open problem posed in [12]. Second, we show that if there exists a homeomorphism $T : C_p(X) \rightarrow E_w$ for some infinite Tychonoff space X and a Banach space E , then (a) X is a countable union of compact sets $X_n, n \in \omega$, where at least one component X_n is non-scattered; (b) E necessarily contains an isomorphic copy of the Banach space ℓ_1 .

1. INTRODUCTION

For a locally convex space E by E_w we denote the space E endowed with the weak topology $w = \sigma(E, E')$ of E , where E' means the topological dual of E . All topological spaces in the paper are assumed to be Tychonoff. For a Tychonoff space X let $C_p(X)$ be the space of continuous real-valued functions $C(X)$ endowed with the pointwise convergence topology. If X is a compact space, then $C(X)$ means the Banach space equipped with the sup-norm.

In [14] M. Krupski asked the following question:

Problem 1.1. *Suppose that K is an infinite (metrizable) compact space. Can $C(K)_w$ and $C_p(K)$ be homeomorphic?*

The main result of [14] shows that the answer is "no", provided K is an infinite metrizable compact C -space (in particular, if K is any infinite metrizable finite-dimensional compact space).

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A more general problem was posed in [15]:

Problem 1.2. *Let K and L be infinite compact spaces. Can it happen that $C_p(L)$ and $C(K)_w$ are homeomorphic?*

Both Problems 1.1, 1.2 remain open, although in some cases the answer is known to be negative. For example, the most easy examples of compact spaces K such that $C_p(K)$ and $C(K)_w$ are not homeomorphic, as already observed in [15, (D), p. 648], are scattered compact spaces. The reason is the following: *the space $C(K)_w$ is not Fréchet-Urysohn for every infinite compact space K* . Recall that a topological space X is called *scattered* if every non-empty subspace of X contains an isolated point, and $C_p(K)$ is Fréchet-Urysohn for any scattered compact space K . We refer the reader to articles [14], [15] (and references therein) discussing these problems and providing more concrete examples of compact spaces K and L such that $C_p(K)$ and $C(L)_w$ are not homeomorphic. We recommend also the paper [16] which surveys a substantial progress in the study of various types of homeomorphisms between the function spaces $C_p(X)$.

One of the starting points of our research is the following result of M. Krupski and W. Marciszewski: for any infinite compact spaces K and L a homeomorphism $T : C(K)_w \rightarrow C_p(L)$ which is in addition uniformly continuous, does not exist (see [15, Proposition 3.1]). In particular, there is no a linear homeomorphism between function spaces $C(K)_w$ and $C_p(L)$ [15, Corollary 3.2].

This short summary makes it clear our motivation for a formulation of the following "linear" variant of the above Problem 1.1.

Problem 1.3 ([12]). *Does there exist an infinite compact space X admitting a continuous linear surjection $T : C_p(X) \rightarrow C(X)_w$?*

It has been proved earlier that no infinite metrizable compact C -space X admits such a mapping [12].

In this note we suggest to consider analogous questions in a more general framework. It was H.H. Corson who initiated the investigation aiming to find criteria or techniques which can be used to determine whether or not a given Banach space E under its weak topology has any of the usual topological properties (see [3]). The next problem combines together two lines of research: both E_w and $C_p(X)$.

Problem 1.4. *Let E be a Banach space which admits certain continuous surjective mappings $T : C_p(X) \rightarrow E_w$ for an infinite Tychonoff space X . 1) Characterize such E ; 2) What are the possible restrictions on X ?*

Thus, this paper continues several lines of research initiated in the recent papers [6], [7], [8], [12], [13], [14], [15].

Recall that a mapping between topological spaces is *sequentially continuous* if it sends converging sequences to converging sequences. The following statement proved in Section 2, which is one of the main results of our paper, gives a negative answer to Problem 1.3 in a very strong form.

Theorem 1.5. *Let X be any Tychonoff space and let E be a Banach space. Then every sequentially continuous linear operator $T : C_p(X) \rightarrow E_w$ has a finite-dimensional range.*

In Section 3 we consider another particular version of Problem 1.4, imposing on T a requirement to be a (non-linear) homeomorphism.

Remark 1.6. Let E be a Banach space, and X be an infinite Tychonoff space. If there is a homeomorphism $T : C_p(X) \rightarrow E_w$, then E cannot be reflexive. This is because E_w is K_σ (a countable union of compact subspaces), for every reflexive Banach space E . However, $C_p(X)$ is K_σ if and only if X is finite [1, Theorem I.2.1].

Our next statement, which is one of the main results of our paper, significantly extends Remark 1.6.

Theorem 1.7. *Let E be a Banach space. If there exist an infinite Tychonoff space X and a homeomorphism $T : C_p(X) \rightarrow E_w$, then*

- (a) X is a countable union of compact sets $X_n, n \in \omega$, where at least one component X_n is non-scattered;
- (b) E contains an isomorphic copy of the Banach space ℓ_1 .

A Banach space E is called *weakly compactly generated* (WCG) if there is a weakly compact subset K in X such that $E = \overline{\text{span}(K)}$. WCG spaces constitute a large and important class of Banach spaces [5]. Every weakly compact subspace of a Banach space is called an *Eberlein compact space*.

Corollary 1.8. *Let E be a WCG (separable) Banach space. If there exist an infinite Tychonoff space X and a homeomorphism $T : C_p(X) \rightarrow E_w$, then*

- (a) X is a countable union of Eberlein (metrizable, respectively) compact spaces $X_n, n \in \omega$, where at least one component X_n is non-scattered;
- (b) E contains an isomorphic copy of the Banach space ℓ_1 .

Corollary 1.9. *Let E be a WCG (separable) Banach space. If there exist an infinite compact space X and a homeomorphism $T : C_p(X) \rightarrow E_w$, then X is a non-scattered Eberlein (metrizable, respectively) compact space and E contains a copy of ℓ_1 .*

Theorem 1.7 provides a generalization of [15, Corollary 5.11, Theorem 5.12], because the Banach space $C(L)$ over a compact space L contains an isomorphic copy of ℓ_1 if and only if L is non-scattered (see [5, Theorem 12.29]). Note that our approach is different from that presented in [15]; we make use of the main result of [18] and some standard arguments from general topology and functional analysis.

Necessary conditions in Theorem 1.7 are not sufficient, because the Banach space ℓ_1 is never homeomorphic to any space $C_p(X)$ (cf. Remark 3.8). This and some other remarks related with the topic were mentioned earlier in [13].

Note that M. Krupski and W. Marciszewski have asked already whether the spaces $C_p(K)$ and $C(K)_w$ are homeomorphic for $K = [0, 1]^\omega$, $K = \beta\mathbb{N}$, $K = \beta\mathbb{N} \setminus \mathbb{N}$ [15, Questions 4.7 - 4.9].

Our notation is standard and follows the book [4]. The closure of a set A is denoted by \overline{A} . The following very well-known notions play an important role in the proof of Theorem 1.7. A topological space X is Fréchet-Urysohn if for each subset $A \subset X$ and each $x \in \overline{A}$ there exists a sequence $\{x_n : n \in \omega\}$ in A which converges to x . A subset B of a topological vector space E is *bounded* in E if for each neighbourhood of zero U in E there exists a scalar λ such that $B \subset \lambda U$.

2. SEQUENTIALLY CONTINUOUS LINEAR MAPPINGS $T : C_p(X) \rightarrow E_w$

Let X be a Tychonoff space. For a function $f : X \rightarrow \mathbb{R}$, we denote the support of f by

$$\text{supp}(f) = \{t \in X : f(t) \neq 0\}.$$

In order to prove Theorem 1.5 we need an elementary lemma.

Lemma 2.1. *Let X be an infinite Tychonoff space and let E be a Banach space. If $T : C_p(X) \rightarrow E_w$ is a sequentially continuous linear operator, then for every sequence $\{U_n : n \in \omega\}$ of pairwise disjoint nonempty open subsets of X there exists N such that $T(f) = 0$ for all $n \geq N$ and $f \in C(X)$ with $\text{supp}(f) \subset U_n$.*

Proof. On the contrary, suppose we can find a strictly increasing sequence $\{k_n : n \in \omega\}$ of natural numbers and a sequence $\{f_n : n \in \omega\} \subset C(X)$ such that $\text{supp}(f_n) \subset U_{k_n}$ and $\|T(f_n)\| > 0$ for every n . Then the sequence $\left\{n \frac{f_n}{\|T(f_n)\|} : n \in \omega\right\}$ pointwise converges to zero, but the sequence of images

$$\left\{n \frac{T(f_n)}{\|T(f_n)\|} : n \in \omega\right\}$$

is not bounded in E . We arrive at a contradiction with the facts that the families of bounded sets in E_w and E coincide and sequentially continuous linear operators map bounded sets into bounded sets. \square

Proof of Theorem 1.5.

First step: X is a compact space.

Denote by A the set of all points t in X such that there exists an open neighbourhood U of t in X such that $T(f) = 0$ for every $f \in C(X)$ satisfying the property $\text{supp}(f) \subset U$. Evidently, the set A is open. We consider the following three cases:

- (1) The set $X \setminus A$ is infinite;
- (2) The set $X \setminus A$ is empty;
- (3) The set $X \setminus A$ is finite and nonempty.

Suppose that the item (1) holds.

Since $X \setminus A$ is infinite, we can find a sequence $\{t_n : n \in \omega\} \subset X \setminus A$ and a sequence $\{U_n : n \in \omega\}$ of pairwise disjoint open subsets of X such that $t_n \in U_n$. Then we choose $f_n \in C(X)$ such that $\text{supp}(f_n) \subset U_n$ and $T(f_n) \neq 0$ for every n . This contradicts Lemma 2.1.

Suppose that the item (2) holds.

Let us fix $f \in C(X)$ for a moment. For every $t \in X$, we can find an open neighbourhood U_t of t in X such that $T(g) = 0$ for every $g \in C(X)$ satisfying $\text{supp}(g) \subset U_t$. Since X is a compact space, we choose a finite set $\{t_1, \dots, t_N\} \subset X$ such that $X = \bigcup_{k=1}^N U_{t_k}$. Now we take the partition of unity subordinated to the open finite cover $\{U_{t_k}\}_{k=1}^N$ [4, p. 300]: the functions $\{g_1, \dots, g_N\} \subset C(X)$ such that

$$\text{supp}(g_k) \subset U_{t_k}, \quad g_k(X) \subset [0, 1], \quad \sum_{j=1}^N g_j(t) = 1$$

for all $1 \leq k \leq N$ and $t \in X$. Note that $\text{supp}(fg_k) \subset U_{t_k}$ for every k , therefore,

$$T(f) = T\left(f\left(\sum_{k=1}^N g_k\right)\right) = \sum_{k=1}^N T(fg_k) = 0.$$

Consequently, $T(f) = 0$ for every $f \in C(X)$, i.e. the range of T is trivial.

Suppose that the item (3) holds.

Since $X \setminus A$ is finite, there exists a continuous linear extension operator $L : C_p(X \setminus A) \rightarrow C_p(X)$ such that $L(f)|_{X \setminus A} = f$ for every $f \in C(X \setminus A)$. Let us fix $f \in C(X)$ for a moment. For every n , we can find open sets V_n and W_n such that

$$X \setminus A \subset V_n \subset \overline{V_n} \subset W_n$$

and

$$|(f - L(f|_{X \setminus A}))(t)| < \frac{1}{n}$$

for every $t \in W_n$. For every $t \in X \setminus W_n$, we choose an open neighbourhood $U_{t,n}$ of t such that $U_{t,n} \subset X \setminus \overline{V_n}$ and $T(g) = 0$ for every $g \in C(X)$ satisfying $\text{supp}(g) \subset U_{t,n}$. Since $X \setminus W_n$ is a compact space, we find a finite set

$$\{t_{1,n}, \dots, t_{N_n,n}\} \subset X \setminus W_n$$

such that

$$X \setminus W_n \subset \bigcup_{k=1}^{N_n} U_{t_{k,n}} \subset X \setminus \overline{V_n}.$$

By standard arguments, we find a partition of unity $\{g_{1,n}, \dots, g_{N_n,n}\} \subset C(X)$ such that

$$\text{supp}(g_{k,n}) \subset U_{t_{k,n}}, \quad g_{k,n}(X) \subset [0, 1], \quad \sum_{j=1}^{N_n} g_{j,n}(t) = 1, \quad \sum_{j=1}^{N_n} g_{j,n}(s) \leq 1$$

for all n , $1 \leq k \leq N_n$, $t \in X \setminus W_n$ and $s \in X$. Therefore,

$$\left\| (f - L(f|_{X \setminus A})) - (f - L(f|_{X \setminus A})) \left(\sum_{k=1}^{N_n} g_{k,n} \right) \right\| \leq \frac{1}{n}$$

and

$$T \left((f - L(f|_{X \setminus A})) \left(\sum_{k=1}^{N_n} g_{k,n} \right) \right) = \sum_{k=1}^{N_n} T \left((f - L(f|_{X \setminus A})) g_{k,n} \right) = 0$$

for every n . It is clear that the sequence

$$\left\{ (f - L(f|_{X \setminus A})) \left(\sum_{k=1}^{N_n} g_{k,n} \right) : n \in \omega \right\}$$

converges to $(f - L(f|_{X \setminus A}))$ in $C_p(X)$. Finally we deduce that

$$T(f) = T(L(f|_{X \setminus A}))$$

for every $f \in C(X)$, i.e. the dimension of the range of T does not exceed the finite size of $X \setminus A$.

Second step: X is any Tychonoff space.

Denote by $C_p^*(X)$ the linear subspace of $C_p(X)$ consisting of all bounded continuous functions on X . Recall a well known fact that $C_p^*(X)$ is sequentially dense in $C_p(X)$. Indeed, if f is any function in $C(X)$, then for each natural n define $f_n \in C_p^*(X)$ by the rule: $f_n(x) = f(x)$ if $|f(x)| \leq n$; $f_n(x) = n$ if $f(x) \geq n$; $f_n(x) = -n$ if $f(x) \leq -n$. Clearly, $\{f_n : n \in \omega\} \subset C_p^*(X)$ and the sequence $\{f_n : n \in \omega\}$ pointwise converges to f .

Every function from $C_p^*(X)$ uniquely extends to a function from $C(\beta X)$, where βX is the Stone-Ćech compactification of X . Denote by π the linear continuous operator of restriction: $\pi : C_p(\beta X) \rightarrow C_p(X)$. The range of π is the linear space $C_p^*(X)$. Now we consider the composition

$$T \circ \pi : C_p(\beta X) \rightarrow E_w.$$

Let C be the range of the operator $T \circ \pi$. Since $T \circ \pi$ is a sequentially continuous linear operator, and βX is compact, the range C is finite-dimensional by the first step. But C coincides with the image $T(C_p^*(X))$. Since $C_p^*(X)$ is sequentially dense in $C_p(X)$ and T is sequentially continuous, we get that C is dense in the range of the operator T . However, C is finite-dimensional, hence complete, finally we conclude that the whole range of T coincides with the finite-dimensional linear space C . \square

3. WHEN $C_p(X)$ AND E_w ARE HOMEOMORPHIC?

Let S be the convergent sequence, that is, the space homeomorphic to $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. It is known that for every compact metrizable space X there exists a continuous surjection $T : C_p(S) \rightarrow C_p(X)$ [10, Remark 3.4]. To keep our paper self-contained we recall the argument in the proof of Proposition 3.1 below. A topological space X is called *analytic* if X is a continuous image of the space $\mathbb{N}^{\mathbb{N}}$, which in turn is homeomorphic to the space of irrationals $J \subset \mathbb{R}$ (see e.g. [19]).

Proposition 3.1. *A locally convex space E is analytic if and only if E is a continuous image of the space $C_p(S)$.*

Proof. Assume first that such a continuous mapping from $C_p(S)$ onto E exists. Since $C_p(S)$ is separable and metrizable and $K_{\sigma\delta}$, hence analytic, the image E is analytic as well. Conversely, assume that E is analytic. We fix a continuous surjection $\varphi : \mathbb{N}^{\mathbb{N}} \rightarrow E$. The space $C_p(S)$ is a $K_{\sigma\delta}$ -subset of \mathbb{R}^S but not K_{σ} . Hence, from the Hurewicz theorem (see [19, Theorem 3.5.4]) it follows that $C_p(S)$ contains a closed copy J of the space of irrationals. Now we apply the classic Dugundji extension theorem (see [17, Theorem 2.2]), to get a continuous surjective mapping $T : C_p(S) \rightarrow E$ extending the mapping φ . \square

Corollary 3.2. *Let E be a Banach space. Then there exists a continuous surjective mapping $T : C_p(S) \rightarrow E_w$ if and only if E is separable.*

Proof. If E is a separable Banach space, then E_w is analytic as a continuous image of a Polish space E . It follows from Proposition 3.1 that there exists a continuous surjection $T : C_p(S) \rightarrow E_w$. Conversely, analytic space is separable, hence E_w and E are separable as well. \square

Let X be an infinite Tychonoff space, and E be a Banach space. For every continuous mapping $T : C_p(X) \rightarrow E_w$ we define the set $B(T)$ as follows: $B(T)$ consists of all points t in X such that there exists an open neighbourhood U of t in X with the property

$$\sup\{\|T(f)\| : f \in C(X), \text{supp}(f) \subset U\} < \infty.$$

Evidently, the set $B(T)$ is open.

The following lemma will be used below.

Lemma 3.3. *Let X be an infinite Tychonoff space and let E be a Banach space. If $T : C_p(X) \rightarrow E_w$ is a sequentially continuous map, then $X \setminus B(T)$ is finite.*

Proof. On the contrary, suppose that the set $X \setminus B(T)$ is infinite.

Claim: *There exist a sequence $\{t_n : n \in \omega\} \subset X \setminus B(T)$ and a sequence $\{U_n : n \in \omega\}$ of pairwise disjoint open subsets of X such that $t_n \in U_n$ for each natural n .*

Indeed, let us take any $s_1, s_2 \in X \setminus B(T)$ such that $s_1 \neq s_2$. We find disjoint open neighbourhoods V_1 and V_2 of s_1 and s_2 , respectively. By the regularity of X ,

we find an open set W_1 such that $s_1 \in W_1 \subset \overline{W_1} \subset V_1$. At least one of the sets $(X \setminus B(T)) \cap V_1$ and $(X \setminus B(T)) \cap (X \setminus \overline{W_1})$ is infinite. If the set $(X \setminus B(T)) \cap V_1$ is infinite, we put $t_1 = s_2$, $U_1 = V_2$ and $A_1 = V_1$. If the set $(X \setminus B(T)) \cap V_1$ is finite and the set

$$(X \setminus B(T)) \cap (X \setminus \overline{W})$$

is infinite, we put $t_1 = s_1$, $U_1 = W_1$ and $A_1 = X \setminus \overline{W_1}$.

Suppose that for some natural n we can find open sets A_n, U_1, \dots, U_n and points $t_1, \dots, t_n \in X \setminus B(T)$ such that the set $(X \setminus B(T)) \cap A_n$ is infinite, $t_k \in U_k$, $A_n \cap U_k = \emptyset$ and $U_m \cap U_j = \emptyset$ for all $1 \leq k \leq n$ and $1 \leq m < j \leq n$. We take any

$$s_{2n+1}, s_{2n+2} \in (X \setminus B(T)) \cap A_n$$

such that $s_{2n+1} \neq s_{2n+2}$. We have disjoint open neighbourhoods V_{2n+1} and V_{2n+2} of s_{2n+1} and s_{2n+2} , respectively.

By the regularity of X , we find an open set W_{n+1} such that

$$s_{2n+1} \in W_{n+1} \subset \overline{W_{n+1}} \subset A_n \cap V_{2n+1}.$$

At least one of the sets

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

and

$$(X \setminus B(T)) \cap A_n \cap (X \setminus \overline{W_{n+1}})$$

is infinite. If the set

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

is infinite, we put $t_{n+1} = s_{2n+2}$, $U_{n+1} = V_{2n+2}$ and $A_{n+1} = A_n \cap V_{2n+1}$. If the set

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

is finite and the set

$$(X \setminus B(T)) \cap A_n \cap (X \setminus \overline{W_{n+1}})$$

is infinite, we put $t_{n+1} = s_{2n+1}$, $U_{n+1} = W_{n+1}$ and $A_{n+1} = A_n \cap (X \setminus \overline{W_{n+1}})$. An appeal to the mathematical induction completes the proof of the Claim.

For every natural n , we find $f_n \in C_p(X)$ such that $\text{supp}(f_n) \subset U_n$ and $\|T(f_{n+1})\| > \|T(f_n)\| + 1$. Then the sequence $\{f_n : n \in \omega\}$ pointwise converges to zero, hence $\{T(f_n) : n \in \omega\}$ converges in E_w , but the sequence $\{\|T(f_n)\| : n \in \omega\}$ is not bounded in E . We arrive at a contradiction with the facts that the families of bounded sets in E_w and E coincide and sequentially continuous functions map convergent sequences into convergent sequences. \square

The above results apply to get the following

Proposition 3.4. *Let S be the convergent sequence and let E be an infinite-dimensional separable Banach space. Then there is no a continuous linear surjection $T : C_p(S) \rightarrow E_w$ but there exists a continuous (non-linear) surjection $T : C_p(S) \rightarrow E_w$ such that $S \setminus B(T)$ is finite.*

Proof. The first claim is an immediate consequence of Theorem 1.5. The second claim is an immediate consequence of Corollary 3.2 and Lemma 3.3. \square

A mapping T in Corollary 3.2 is never a homeomorphism, because $C_p(S)$ is a metrizable and infinite-dimensional locally convex space, while E_w is metrizable provided it is finite-dimensional. In this section we are interested in finding necessary conditions for the existing of a homeomorphism $T : C_p(X) \rightarrow E_w$. In order to find such conditions it is reasonable first to examine several basic topological properties that are satisfied by all infinite-dimensional spaces E_w .

Remark 3.5.

(1) *Countable chain condition (ccc).*

It is a fundamental result on the weak topology, due to H.H. Corson, that E_w satisfies the ccc property for every Banach space E [3, proof of Lemma 5]. However, $C_p(X)$ also always enjoys the ccc property [1, Corollary 0.3.7], so we cannot distinguish these spaces by ccc.

(2) *Angelicity.*

Another fundamental result about weak topology is the Eberlein-Šmulian theorem for every space E_w . However, $C_p(X)$ also always enjoys this property for every compact space X [1], so we cannot distinguish much E_w and $C_p(X)$ by angelicity.

(3) *Eberlein-Grothendieck property.*

A topological space Z is called an *Eberlein-Grothendieck space* if Z homeomorphically embeds into the space $C_p(K)$ for some compact space K (see [1, p. 95]). It is widely known that E_w always embeds into $C_p(K)$, where K is the compact unit ball of the dual E' endowed with the weak* topology, i.e. E_w always is an Eberlein-Grothendieck space.

(4) *k-space, sequentiality, Fréchet-Urysohn property.*

All the three properties coincide for each $C_p(X)$ by the Gerlits-Nagy theorem (see [1]). If X is compact then $C_p(X)$ enjoys these properties if and only if X is scattered. Vice versa, if E_w is Fréchet-Urysohn, then E is finite-dimensional. Several alternative proofs are known for this statement. a) By [11, Lemma 14.6] the closed unit ball B in E is a w -neighbourhood of zero, so E is finite-dimensional; b) M. Krupski and W. Marciszewski [15, Corollary 6.5] gave a simple proof for a stronger statement: E_w is not a k -space, if E is an infinite-dimensional Banach space; and c) Original proof of the latter fact appears in [20, p. 280].

We need several auxiliary results.

Lemma 3.6. *Let a Tychonoff space X can be represented as a countable union of scattered compact sets. Then the space $C_p(X)$ is Fréchet-Urysohn.*

Proof. Denote $X = \bigcup \{X_n : n \in \omega\}$, where each X_n is a scattered compact space. Define Y_n to be the \aleph_0 -modification of the topological space X_n , i.e. the family of all G_δ -sets of X_n is declared as a base of the topology of Y_n . It is known that

every Y_n is a Lindelöf P -space (see e.g. [1, Lemma II.7.14]). Define Y to be the free countable union of all Y_n . Evidently, Y remains a Lindelöf P -space. We define a natural continuous mapping φ from Y onto X as follows. Let $y \in Y$, then $y = x \in Y_n$ for a certain unique $n \in \omega$, and we define $\varphi(y) = x \in X$. The map dual to φ homeomorphically embeds $C_p(X)$ into $C_p(Y)$. The space $C_p(Y)$ is Fréchet-Urysohn (see e.g. [1, Theorem II.7.15]), therefore the space $C_p(X)$ is Fréchet-Urysohn as well. \square

Lemma 3.7. *Let X be a non-scattered compact space. Then for every finite set $A \subset X$ there is a non-scattered compact set Y such that $Y \subset X \setminus A$.*

Proof. We shall use the following classic theorem of A. Pełczyński and Z. Semadeni: Let X be a compact space, then X is scattered if and only if there is no continuous mapping of X onto the segment $[0, 1]$ (see [21, Theorem 8.5.4]). Since X is not scattered, there exists a continuous surjection $f : X \rightarrow [0, 1]$. Since the set A is finite, we find a segment $[a, b] \subset [0, 1] \setminus f(A)$. Then $Y = f^{-1}([a, b])$ is a compact non-scattered subset of X . \square

Now we are ready to present the proof of Theorem 1.7.

Proof of Theorem 1.7. We have already observed in Remark 3.5 that E_w always is an Eberlein-Grothendieck space. Hence, by our assumptions $C_p(X)$ is the image under a continuous open mapping of an Eberlein-Grothendieck space. Making use of the fundamental result of O. Okunev, we immediately conclude that X must be a σ -compact space (see [18, Theorem 4] or [1, Corollary III.2.9]).

Denote $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is a compact space. We claim that at least one component X_n is non-scattered, and then clearly X is non-scattered. Indeed, otherwise, the space $C_p(X)$ would be Fréchet-Urysohn by Lemma 3.6, therefore also E_w would be Fréchet-Urysohn, which is false, again by Remark 3.5. Fix any n such that the compact set X_n is non-scattered.

According to Lemma 3.3, the set $X \setminus B(T)$ is finite, therefore we can apply Lemma 3.7 and find a non-scattered compact set $Y \subset X_n \cap B(T)$. For every $t \in Y$, we choose open sets V_t and W_t such that $t \in V_t \subset \overline{V_t} \subset W_t \subset B(T)$ and

$$\sup\{\|T(g)\| : g \in C(X), \text{supp}(g) \subset W_t\} < \infty.$$

Since Y is compact, we find finitely many sets V_t covering Y . There exists at least one $t \in Y$ such that $\overline{V_t} \cap Y$ is not scattered. Indeed, assuming that each $\overline{V_t} \cap Y$ is scattered we would get that a non-scattered compact space Y is covered by finitely many scattered compact sets, which is obviously impossible. Fix a non-scattered $Z = \overline{V_t} \cap Y$. The next fact plays a crucial role: the space $C_p(Z)$ is not Fréchet-Urysohn. We prove the following

Claim: $F = \{f \in C(X) : \text{supp}(f) \subset W_t\}$ is not a Fréchet-Urysohn space.

Indeed, it is clear that F is a closed subset of $C_p(X)$. Let G be a subset of $C_p(Z)$ satisfying the property: there exists $g \in \overline{G}$ such that does not exist any sequence

in G which converges to g in $C_p(X)$. Let

$$H = \{f \in F : f|_Z \in G\}.$$

Using the Tietze-Urysohn theorem and the Urysohn lemma we deduce that $\{f|_Z : f \in F\} = C_p(Z)$. Hence the space $\{f|_Z : f \in F\}$ fails the Fréchet-Urysohn property. To complete the proof it is enough to show that $\{f|_Z : f \in \overline{H}\} = \overline{G}$. It is clear that $\{f|_Z : f \in \overline{H}\} \subset \overline{G}$. Suppose that

$$h \in \overline{G} \setminus \{f|_Z : f \in \overline{H}\}.$$

Let $\tilde{h} \in F$ be such that $\tilde{h}|_Z = h$. Since \overline{H} is a closed subset of $C_p(X)$ and $\tilde{h} \notin \overline{H}$, we find $N \in \mathbb{N}$, $\{t_1, \dots, t_N\} \subset W_t$ and $\varepsilon_j > 0$ for every $1 \leq j \leq N$ such that

$$\{f \in F : |\tilde{h}(t_j) - f(t_j)| < \varepsilon_j, 1 \leq j \leq N\} \cap \overline{H} = \emptyset.$$

Either $\{t_1, \dots, t_N\} \cap Z = \emptyset$ or $\{t_1, \dots, t_N\} \cap Z \neq \emptyset$. In the first case, according to the Tietze-Urysohn theorem for every $f \in G$ we find $\tilde{f} \in H$ such that $\tilde{f}|_Z = f$ and $\tilde{f}(t_j) = \tilde{h}(t_j)$ for every $1 \leq j \leq N$. Consequently, only the second case may hold. We may assume that there exists $1 \leq L \leq N$ such that

$$\{t_1, \dots, t_N\} \cap Z = \{t_1, \dots, t_L\}.$$

It is clear that

$$\{f|_Z : f \in F, |h(t_j) - f(t_j)| < \varepsilon_j, 1 \leq j \leq L\} \cap G = \emptyset.$$

Consequently, $h \notin \overline{G}$. Thus we have arrived at a contradiction. The Claim has been proved.

Finally, relying on the definition of F , we observe that $T(F)$ is a bounded subset of E which is not a Fréchet-Urysohn space in the weak topology. According to [2, Proposition 4.4], the Banach space E contains a subspace isomorphic to l_1 , which finishes the proof. \square

In [15, Corollary 5.11, Theorem 5.12] M. Krupski and W. Marciszewski proved that for infinite compact spaces K and L , where L is scattered, the spaces $C_p(K)$, $C(L)_w$ and the spaces $C_p(L)$, $C(K)_w$ are not homeomorphic. Our Theorem 1.7 generalizes both results because the Banach space $C(L)$ does not contain a subspace isomorphic to l_1 , if L is scattered.

Proof of Corollary 1.8. By Theorem 1.7 we have that $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is a compact space. On the other hand, if E is a WCG Banach space, then E_w contains a dense σ -compact subspace. Therefore, $C_p(X)$ also contains a dense σ -compact subspace, which we denote by Y . For each $n \in \omega$ consider the restriction mapping π_n from $C_p(X)$ onto $C_p(X_n)$. It is easily seen that $Z_n = \pi_n(Y)$ is a dense σ -compact subspace of $C_p(X_n)$. It follows that a compact space X_n is Eberlein (see [1, Theorem IV.1.7]) for each n . Assume now that E is separable, then E_w is also separable, hence analogously to the above case, $C_p(X_n)$ is separable

for each n . It follows that a compact space X_n is metrizable (see [1, Theorem I.1.5]) for each n . The rest is provided by Theorem 1.7. \square

Proof of Corollary 1.9. This is done actually in the previous proof. Just replace X_n by X . \square

Remark 3.8. Necessary conditions in Theorem 1.7 are not sufficient, because the Banach space $E = \ell_1$ in the weak topology is never homeomorphic to any space $C_p(X)$. The reason is the following: every separable Banach space E with the Schur property is an \aleph_0 -space in the weak topology. But $C_p(X)$ is an \aleph_0 -space if and only if X is countable, i.e. if and only if $C_p(X)$ is metrizable. (For the details see [7]).

Remark 3.9. The arguments used in the proof of Theorem 1.7 are not applicable for the question whether X in that result must be a compact space and not just a σ -compact space. This is because compactness of X is not invariant under the homeomorphisms of the spaces $C_p(X)$. For instance, the spaces $C_p[0, 1]$ and $C_p(\mathbb{R})$ are homeomorphic [9].

We finish the paper by the following challenging question.

Problem 3.10. *Does there exist a separable Banach space E such that E_w is homeomorphic to $C_p[0, 1]$?*

REFERENCES

- [1] A. V. Arkhangel'ski, *Topological Function Spaces*, Kluwer, Dordrecht, 1992.
- [2] C. S. Barroso, O. F. K. Kalenda, P. K. Lin, *On the approximate fixed point property in abstract spaces*, *Math. Z.* **271** (2012), 1271–1285.
- [3] H. H. Corson, *The weak topology of a Banach space*, *Trans. Amer. Math. Soc.* **101** (1961), 1–15.
- [4] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
- [5] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books Math./Ouvrages Math. SMC, 2001.
- [6] S. Gabrielyan, J. Grebik, J. Kąkol, L. Zdomskyy, *The Ascoli property for function spaces*, *Topology Appl.* **214** (2016), 35–50.
- [7] S. Gabrielyan, J. Kąkol, W. Kubiś, W. Marciszewski, *Networks for the weak topology of Banach and Fréchet spaces*, *J. Math. Anal. Appl.* **432** (2015), 1183–1199.
- [8] S. Gabrielyan, J. Kąkol, G. Plebanek, *The Ascoli property for function spaces and the weak topology on Banach and Fréchet spaces*, *Studia Math.* **233** (2016), 119–139.
- [9] S. P. Gul'ko, T. E. Khmyleva, *Compactness is not preserved by the relation of t -equivalence*, *Mathematical Notes of the Academy of Sciences of the USSR*, **39** (1986), 484–488.
- [10] K. Kawamura, A. Leiderman, *Linear continuous surjections of C_p -spaces over compacta*, *Topology Appl.* **227** (2017), 135–145.
- [11] J. Kąkol, W. Kubiś, M. Lopez-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, *Developments in Mathematics*, Springer, New York, 2011.
- [12] J. Kąkol, A. Leiderman, *On linear continuous operators between distinguished spaces $C_p(X)$* , to appear in RACSAM.

- [13] J. Kąkol, S. Moll-López, *A note on the weak topology of spaces $C_k(X)$ of continuous functions*, RACSAM (2021) 115:125, <https://doi.org/10.1007/s13398-021-01051-1>
- [14] M. Krupski, *On the weak and pointwise topologies in function spaces*, RACSAM **110** (2016), 557–563.
- [15] M. Krupski, W. Marciszewski, *On the weak and pointwise topologies in function spaces II*, J. Math. Anal. Appl. **452** (2017), 646–658.
- [16] W. Marciszewski, *Function Spaces*, in Recent Progress in General Topology II, Edited by M. Hušek, J. van Mill, North-Holland (2002), 345–369.
- [17] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland Mathematical Library **64**, North-Holland, Amsterdam, 2001.
- [18] O. G. Okunev, *Weak topology of an associated space and t -equivalence*, Mathematical Notes of the Academy of Sciences of the USSR, **46** (1989), 534–538.
- [19] C. A. Rogers, J. E. Jayne, *K -analytic sets*, in: Analytic Sets, Academic Press, 1980, p. 1–181.
- [20] G. Schlüchtermann, R. F. Wheeler, *The Mackey dual of a Banach space*, Note di Mat. **11** (1991), 273–287.
- [21] Z. Semadeni, *Banach spaces of continuous functions*, Volume I, PWN - Polish Scientific Publishers, Warszawa, 1971.

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