

# THE KERNEL GENERATING CONDITION AND ABSOLUTE GALOIS GROUPS

IDO EFRAT

ABSTRACT. For a list  $\mathcal{L}$  of finite groups and for a profinite group  $G$ , we consider the intersection  $T(G)$  of all open normal subgroups  $N$  of  $G$  with  $G/N$  in  $\mathcal{L}$ . We give a cohomological characterization of the epimorphisms  $\pi: S \rightarrow G$  of profinite groups (satisfying some additional requirements) such that  $\pi[T(S)] = T(G)$ . For  $p$  prime, this is used to describe cohomologically the profinite groups  $G$  whose  $n$ th term  $G_{(n,p)}$  (resp.,  $G^{(n,p)}$ ) in the  $p$ -Zassenhaus filtration (resp., lower  $p$ -central filtration) is an intersection of this form. When  $G = G_F$  is the absolute Galois group of a field  $F$  containing a root of unity of order  $p$ , we recover as special cases results by Mináč, Spira and the author, describing  $G_{(3,p)}$  and  $G^{(3,p)}$  as  $T(G)$  for appropriate lists  $\mathcal{L}$ .

## 1. INTRODUCTION

Given a field  $F$ , let  $G_F = \text{Gal}(F^{\text{sep}}/F)$  be its absolute Galois group, considered as a profinite group. It is a major open problem in modern Galois theory to know which profinite groups  $G$  are realizable as  $G_F$  for some field  $F$ . Throughout this paper we fix a prime number  $p$  and assume that the field  $F$  contains a root of unity of order  $p$ , and in particular  $\text{char } F \neq p$ . Among the few known group-theoretic restrictions on the structure of  $G = G_F$  are several “Intersection Properties” related to standard filtrations of  $G$  by characteristic subgroups: The *lower  $p$ -central filtration*  $G^{(n,p)}$ , and the  *$p$ -Zassenhaus filtration*  $G_{(n,p)}$ ,  $n = 1, 2, \dots$ . We recall that they are defined inductively (in the profinite sense) by

$$G^{(1,p)} = G, \quad G^{(n+1,p)} = (G^{(n,p)})^p [G, G^{(n,p)}],$$

$$G_{(1,p)} = G, \quad G_{(n,p)} = (G_{(\lceil n/p \rceil, p)})^p \prod_{i+j=n} [G_{(i,p)}, G_{(j,p)}] \text{ for } n \geq 2.$$

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By results by Mináč, Spira and the author, for an absolute Galois group  $G = G_F$  as above, some of these filtration subgroups can be expressed as the intersection of the open subgroups  $N$  of  $G$  such that  $G/N$  belongs to a certain list  $\mathcal{L}$  of finite groups. Namely:

- (i) When  $p = 2$ ,  $G^{(3,2)} = \bigcap \{N \trianglelefteq G \mid G/N \cong \mathbb{Z}/2, \mathbb{Z}/4, D_4\}$  [MSp96, Cor. 2.18];
- (ii) When  $p > 2$ ,  $G^{(3,p)} = \bigcap \{N \trianglelefteq G \mid G/N \cong \mathbb{Z}/p^2, M_{p^3}\}$  [EfMi11];
- (iii) For every  $p$ ,  $G_{(3,p)} = \bigcap \{N \trianglelefteq G \mid G/N \cong \mathbb{Z}/p, \mathbb{U}_2(\mathbb{Z}/p)\}$  [EfMi17, Th. D].

Here  $D_4$  is the dihedral group of order 8,  $M_{p^3}$  is the extra-special group of order  $p^3$  and exponent  $p^2$  (see §11), and  $\mathbb{U}_n(R)$  denotes the group of all  $(n+1) \times (n+1)$  unipotent upper-triangular matrices over a ring  $R$ .

Furthermore, when  $G$  is a free profinite group and for arbitrary  $n \geq 2$ , one has

- (iv)  $G_{(n,p)} = \bigcap \{N \leq G \mid G/N \leq \mathbb{U}_n(\mathbb{Z}/p)\}$  [Efr14a, Th. A].

Mináč and Tân [MT15] conjectured that (iv) should hold more generally for every absolute Galois group  $G = G_F$  as above.

It should be stressed that the intersection theorems (i)–(iii) do not hold for general profinite groups. Their proofs rely on deep cohomological properties of absolute Galois groups, in particular the behavior of special elements in  $H^2(G_F, \mathbb{Z}/p)$ , such as cup products, Bockstein elements, and elements of  $n$ -fold Massey products.

While the known proofs of these intersection theorems are rather different one from each other, they share a common general structure: First one proves the equality

$$(1.1) \quad G^{(n,p)} = \bigcap \{N \trianglelefteq G \mid G/N \in \mathcal{L}\} \quad (\text{or } G_{(n,p)} = \bigcap \{N \trianglelefteq G \mid G/N \in \mathcal{L}\})$$

when  $G$  is a free profinite group. Then, for a more general profinite group  $G$  (such as  $G_F$ ), one takes a profinite presentation, i.e., a continuous epimorphism  $\pi: S \rightarrow G$ , where  $S$  is a free profinite group, and transfers the equality (1.1) from  $S$  to  $G$ .

The first part is purely group-theoretic, and is usually proved using *Magnus theory*, i.e., by viewing the elements of  $G = S$  as formal power series. A general machinery to obtain such results in the free profinite case is given in [Efr14b] (see also [CE16]).

In the current paper we focus on the second part, and prove a general cohomological *Transfer Principle*, which allows one to transfer equalities as in (1.1) from a group  $S$  to its quotient  $G$ . To explain it, we note that

for an epimorphism  $\pi: S \rightarrow G$ , we clearly have  $\pi[S^{(n,p)}] = G^{(n,p)}$  (resp.,  $\pi[S_{(n,p)}] = G_{(n,p)}$ ). However, in general, one does not have that

$$(1.2) \quad \pi \left[ \bigcap \{N \trianglelefteq S \mid S/N \in \mathcal{L}\} \right] = \bigcap \{M \trianglelefteq G \mid G/M \in \mathcal{L}\}.$$

Our transfer principle gives a cohomological condition on special elements of  $H^2(G, \mathbb{Z}/p)$  which is equivalent to the validity of (1.2). From this principle, one can deduce as special cases the intersection theorems (i)–(iii) of [MSp96], [EfMi11] and [EfMi17], respectively, as well as several main results of [Efr14a].

For the transfer principle, we consider intersections  $T(G)$  and  $\bar{T}(G)$  of certain closed normal subgroups of  $G$  (see below) with  $T(G) \leq \bar{T}(G)$ , and an associated collection  $\Pi$  of “special cohomology elements” in  $H^2(G/\bar{T}(G))$ , where  $H^i(\bar{G}) = H^i(\bar{G}, \mathbb{Z}/p)$  denotes the  $i$ th profinite cohomology group of the profinite group  $\bar{G}$  with the trivial action on  $\mathbb{Z}/p$ . We further consider the inflation map  $\text{inf}: H^2(G/\bar{T}(G)) \rightarrow H^2(G)$ . For a subset  $\Pi_0$  of  $H^2(G/\bar{T}(G))$  let  $\langle \Pi_0 \rangle$  be the subgroup it generates.

**The Transfer Theorem.** *Let  $\pi: S \rightarrow G$  be a continuous homomorphism of profinite groups, where  $\text{Ker}(\pi) \leq \bar{T}(S)$ . The following conditions are equivalent:*

- (a)  $\pi[T(S)] = T[G]$ ;
- (b)  $\langle \Pi \cap \text{Ker}(\text{inf}) \rangle = \langle \Pi \rangle \cap \text{Ker}(\text{inf})$ .

More specifically, the intersections  $T(G)$ ,  $\bar{T}(G)$  and the set  $\Pi$  are as follows: We consider a collection of central extensions

$$\omega: \quad 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{U}_\omega \rightarrow \bar{\mathbb{U}}_\omega \rightarrow 1$$

of profinite groups subject to certain assumptions (see §3). Let  $\alpha_\omega$  be the classifying element of  $\omega$  in  $H^2(\bar{\mathbb{U}}_\omega)$ . For a profinite group  $G$  we define  $T(G)$  (resp.,  $\bar{T}(G)$ ) to be the intersection of all kernels of continuous homomorphisms  $\rho: G \rightarrow \mathbb{U}_\omega$  (resp.,  $\rho: G \rightarrow \bar{\mathbb{U}}_\omega$ ) for some  $\omega$ . The set  $\Pi$  of special cohomology elements consists of the pullbacks  $\bar{\rho}^*(\alpha_\omega)$  of  $\alpha_\omega$  to  $H^2(G/\bar{T}(G))$  for some  $\omega$  and some continuous homomorphism  $\bar{\rho}: G/\bar{T}(G) \rightarrow \bar{\mathbb{U}}_\omega$  which lifts to a continuous homomorphism  $\rho: S \rightarrow \mathbb{U}_\omega$ . Note that when  $S$  is free, the latter condition is always satisfied. For certain natural choices of  $\omega$ , these pullbacks yield the cup products, and more generally, Massey product elements, and Bockstein elements (§9, §11). See Theorem 7.5 and Remark 7.6.

The paper is organized as follows: In §2 we assemble some needed facts on bilinear maps. In §3–§7 we develop a formal group-theoretic apparatus

which eventually leads to the general Transfer Theorem (Theorem 7.5). It is based on Hoechsmann’s theory [Hoe68], which connects lifting of homomorphisms in group extensions to the transgression map in cohomology. This is combined with delicate computations in Pontrjagin duality, which interpret the profinite group side (as in (a)) to the cohomology side (as in (b)). Then we apply the general theory in three situations:

- (1) The *p-Zassenhaus context*, where the central extension arises from  $\mathbb{U}_n(\mathbb{Z}/p)$  and the special cohomology elements are  $n$ -fold Massey product elements (§9);
- (2) The *lower p-central context*, where the central extensions arise from  $\mathbb{U}_s(\mathbb{Z}/p^{n-s+1})$ ,  $s = 1, 2, \dots, n$ , and the special cohomology elements are the *unitriangular spectrum* studied in [Efr17], [Efr20a] (§10);
- (3) A “*mixed*” context, where the central extensions arise from  $\mathbb{Z}/p^2$  and  $M_{p^3}$ ,  $p > 2$ , and the special cohomology elements are Bockstein elements and sums of Bockstein elements and cup products (§11).

In each context we obtain a particular cohomological transfer theorem, and moreover, extend earlier results from [Efr14a], [CEM12], [EfMi17], and [MSp96]. In §12 we apply the results in these three contexts to recover the intersection theorems (i)–(iii) in a uniform manner. This part uses deep cohomological properties of absolute Galois groups: The injectivity part of the Merkurjev–Suslin theorem and the description of the Bockstein map as a cup product. Finally, in §13 we give examples which show that two seemingly plausible variants of the intersection theorems fail to hold. §§12–13 are closely related at many points to the papers [EfMi11] and [EfMi17], and I thank Ján Mináč for the collaboration on these works.

## 2. BILINEAR MAPS

Let  $A$  be a profinite abelian group,  $B$  a discrete abelian group, and  $Z$  a finite abelian group. Let  $(\cdot, \cdot): A \times B \rightarrow Z$  be a bilinear map. It induces homomorphisms

$$A \rightarrow \text{Hom}(B, Z), \quad B \rightarrow \text{Hom}_{\text{cont}}(A, Z).$$

We refer to their kernels as the *left kernel* (resp., the *right kernel*) of  $(\cdot, \cdot)$ . We say that  $(\cdot, \cdot)$  is *left-surjective* (resp., *right-surjective*) if the left (resp., right) induced homomorphism is surjective. The map  $(\cdot, \cdot)$  is *non-degenerate* if both induced homomorphisms are injective, and is *perfect* if they are both isomorphisms.

Let  $\alpha: A_1 \rightarrow A_2$  be a continuous homomorphism of profinite abelian groups and  $\beta: B_2 \rightarrow B_1$  a homomorphism of discrete abelian groups. Suppose that we have a diagram of bilinear maps

$$(2.1) \quad \begin{array}{ccc} A_1 & \times & B_1 \xrightarrow{(\cdot, \cdot)_1} Z \\ \alpha \downarrow & & \uparrow \beta \quad \parallel \\ A_2 & \times & B_2 \xrightarrow{(\cdot, \cdot)_2} Z \end{array}$$

which is commutative, in the sense that  $(a, \beta(b))_1 = (\alpha(a), b)_2$  for every  $a \in A_1$  and  $b \in B_2$ .

**Lemma 2.1.** (a) *The commutative diagram (2.1) induces a bilinear map  $(\cdot, \cdot)_{\text{Coker}(\alpha), \text{Ker}(\beta)}: \text{Coker}(\alpha) \times \text{Ker}(\beta) \rightarrow Z$ .*

(b) *If the right kernel of  $(\cdot, \cdot)_2$  is trivial, then the right kernel of  $(\cdot, \cdot)_{\text{Coker}(\alpha), \text{Ker}(\beta)}$  is trivial.*

(c) *If the natural map  $\pi: A_2 \rightarrow \text{Coker}(\alpha)$  splits, the right kernel of  $(\cdot, \cdot)_1$  is trivial, and  $(\cdot, \cdot)_2$  is right-surjective, then  $(\cdot, \cdot)_{\text{Coker}(\alpha), \text{Ker}(\beta)}$  is right-surjective.*

*Proof.* For  $a' \in A_2$  and  $b \in \text{Ker}(\beta)$  we set  $(\pi(a'), b)_{\text{Coker}(\alpha), \text{Ker}(\beta)} = (a', b)_2$  (with  $\pi$  as in (c)). It is well-defined, since for  $a' = \alpha(a)$ ,  $a \in A_1$ , we have  $(a', b)_2 = (a, \beta(b))_1 = (a, 0)_1 = 0$ . It is straightforward to verify the bilinearity, as well as assertion (b).

For (c), we decompose  $A_2 = \text{Im}(\alpha) \oplus A'_2$ , where  $\pi$  maps  $A'_2$  bijectively onto  $\text{Coker}(\alpha)$ . Let  $\lambda_0: \text{Coker}(\alpha) \rightarrow Z$  be a continuous homomorphism. Let  $\lambda: A_2 \rightarrow Z$  be the continuous homomorphism which is 0 on  $\text{Im}(\alpha)$  and is  $\lambda_0 \circ \pi$  on  $A'_2$ . By assumption, there exists  $b \in B_2$  such that  $\lambda = (\cdot, b)_2$ . For every  $a \in A_1$  we have  $(a, \beta(b))_1 = (\alpha(a), b)_2 = \lambda(\alpha(a)) = 0$ , whence  $b \in \text{Ker}(\beta)$ . Moreover, for every  $a' \in A'_2$  one has

$$\lambda_0(\pi(a')) = \lambda(a') = (a', b)_2 = (\pi(a'), b)_{\text{Coker}(\alpha), \text{Ker}(\beta)},$$

i.e.,  $\lambda_0 = (\cdot, b)_{\text{Coker}(\alpha), \text{Ker}(\beta)}$ , as required.  $\square$

**Corollary 2.2.** *Suppose that  $A_i, B_i$ ,  $i = 1, 2$ , have prime exponent  $p$ , and that  $Z = \mathbb{Z}/p$ . Assume that the right kernel of  $(\cdot, \cdot)_1$  is trivial and that  $(\cdot, \cdot)_2$  is perfect. Then  $(\cdot, \cdot)_{\text{Coker}(\alpha), \text{Ker}(\beta)}$  is perfect.*

*Proof.* Since  $A_2$  is an elementary abelian  $p$ -group, the splitting assumption in Lemma 2.1(c) is satisfied. Hence  $(\cdot, \cdot)_{\text{Coker}(\alpha), \text{Ker}(\beta)}$  is right-surjective. By Lemma 2.1(b), its right kernel is trivial, so the induced map  $\text{Ker}(\beta) \rightarrow \text{Hom}_{\text{cont}}(\text{Coker}(\alpha), \mathbb{Z}/p)$  is an isomorphism. Thus  $\text{Coker}(\alpha)$  and  $\text{Ker}(\beta)$  are Pontrjagin duals, and the assertion follows.  $\square$

**Lemma 2.3.** *Suppose that  $A_i, B_i, i = 1, 2$ , have prime exponent  $p$ , and that  $Z = \mathbb{Z}/p$ . Assume that  $(\cdot, \cdot)_1$  is left-surjective and  $\beta$  is injective. Then  $(\cdot, \cdot)_2$  is left-surjective.*

*Proof.* The assumptions imply that  $B_2$  is a direct factor of  $B_1$ . Hence the dual map  $\beta^\vee: \text{Hom}(B_1, \mathbb{Z}/p) \rightarrow \text{Hom}(B_2, \mathbb{Z}/p)$  is surjective. Therefore in the commutative square

$$\begin{array}{ccc} A_1 & \longrightarrow & \text{Hom}(B_1, \mathbb{Z}/p) \\ \alpha \downarrow & & \downarrow \beta^\vee \\ A_2 & \longrightarrow & \text{Hom}(B_2, \mathbb{Z}/p). \end{array}$$

the lower horizontal map is surjective. □

### 3. THE SUBGROUPS $T$ AND $\bar{T}$

For profinite groups  $\mathbb{U}$  and  $G$ , let

$$T^\mathbb{U}(G) = \bigcap \text{Ker}(\rho),$$

where the intersection is over all continuous homomorphisms  $\rho: G \rightarrow \mathbb{U}$ . Alternatively,  $T^\mathbb{U}(G) = \bigcap N$ , where  $N$  ranges over all closed normal subgroups of  $G$  such that  $G/N$  embeds as a closed subgroup of  $\mathbb{U}$ . The proof of the following lemma is straightforward:

**Lemma 3.1.** *Let  $\mathbb{U}, \mathbb{U}_1, \dots, \mathbb{U}_n$  be profinite groups, and suppose that there are continuous homomorphisms  $\gamma_i: \mathbb{U} \rightarrow \mathbb{U}_i, i = 1, 2, \dots, n$ , such that  $\bigcap_{i=1}^n \text{Ker}(\gamma_i) = \{1\}$ . For every profinite group  $G$  one has  $\bigcap_{i=1}^n T^{\mathbb{U}_i}(G) \leq T^\mathbb{U}(G)$ .*

Let  $Z$  be a finite abelian group. We fix a nonempty set  $\Omega$  of central extensions of profinite groups

$$\omega: 0 \rightarrow Z \rightarrow \mathbb{U} \xrightarrow{\lambda} \bar{\mathbb{U}} \rightarrow 1.$$

We assume that for every  $\omega$ :

- (I) There is a family of continuous homomorphisms  $\gamma_i: \bar{\mathbb{U}} \rightarrow \mathbb{U}, i = 1, 2, \dots, n$ , such that  $\bigcap_{i=1}^n \text{Ker}(\gamma_i) = \{1\}$ .
- (II)  $Z$  embed in  $\bar{\mathbb{U}}$ .

For a profinite group  $G$ , assumption (I) and Lemma 3.1 imply that  $T^\mathbb{U}(G) \leq T^{\bar{\mathbb{U}}}(G)$ . Let

$$T(G) = \bigcap_{\omega \in \Omega} T^{\mathbb{U}_\omega}(G), \quad \bar{T}(G) = \bigcap_{\omega \in \Omega} T^{\bar{\mathbb{U}}_\omega}(G).$$

Then  $T(G)$ ,  $\bar{T}(G)$  are closed normal subgroups of  $G$ , and

$$T(G) \leq \bar{T}(G).$$

If  $\pi: G_1 \rightarrow G_2$  is a continuous homomorphism of profinite groups, then

$$(3.1) \quad \pi[T(G_1)] \leq T(G_2), \quad \pi[\bar{T}(G_1)] \leq \bar{T}(G_2).$$

**Lemma 3.2.** *Let  $\pi: G_1 \rightarrow G_2$  is a continuous epimorphism of profinite groups. Then:*

- (a)  $\text{Ker}(\pi) \leq T(G_1)$  if and only if  $T(G_1) = \pi^{-1}[T(G_2)]$ .
- (b)  $\text{Ker}(\pi) \leq \bar{T}(G_1)$  if and only if  $\bar{T}(G_1) = \pi^{-1}[\bar{T}(G_2)]$ .

*Proof.* (a) The “if” part is immediate.

For the “only if” part, assume that  $\text{Ker}(\pi) \leq T(G_1)$ . Then every continuous homomorphism  $\rho_1: G_1 \rightarrow \mathbb{U}_\omega$ ,  $\omega \in \Omega$ , factors via a continuous homomorphism  $\rho_2: G_2 \rightarrow \mathbb{U}_\omega$ . We have  $\rho_1[\pi^{-1}[T(G_2)]] = \rho_2[T(G_2)] = 1$ . Thus  $\pi^{-1}[T(G_2)] \leq T(G_1)$ . The opposite inclusion follows from (3.1).

(b) Similarly. □

**Lemma 3.3.** *If  $Z$  has exponent  $m$ , then  $\bar{T}(G)^m[G, \bar{T}(G)] \leq T(G)$  for every profinite group  $G$ . Consequently,  $\bar{T}(G)/T(G)$  is an abelian group of exponent dividing  $m$ .*

*Proof.* Let  $\omega \in \Omega$ . Since it is central,  $Z^m[\mathbb{U}_\omega, Z] = 1$ . For every continuous homomorphism  $\rho: G \rightarrow \mathbb{U}_\omega$  we have  $\rho(\bar{T}(G)) \leq \text{Ker}(\lambda_\omega) = Z$ , whence  $\rho(\bar{T}(G)^m[G, \bar{T}(G)]) = 1$ , and the assertion follows. □

For a profinite group  $G$  which acts trivially on  $Z$ , we write  $H^i(G) = H^i(G, Z)$  for the  $i$ th (profinite) cohomology group. Recall that  $H^1(G) = \text{Hom}_{\text{cont}}(G, Z)$ . We write  $\text{inf}$ ,  $\text{res}$  and  $\text{trg}$  for the inflation, restriction, and transgression homomorphisms, respectively [NSW08, Ch. I].

**Lemma 3.4.** *Let  $N$  be a closed normal subgroup of  $G$  contained in  $\bar{T}(G)$ .*

- (a) *The map  $\text{inf}: H^1(G/N) \rightarrow H^1(G)$  is an isomorphism.*
- (b) *There is an exact sequence*

$$0 \rightarrow H^1(N)^G \xrightarrow{\text{trg}} H^2(G/N) \xrightarrow{\text{inf}} H^2(G).$$

*Proof.* (a) Let  $\rho: G \rightarrow Z$  be a continuous homomorphism. By assumption (II),  $Z$  embeds in every  $\bar{\mathbb{U}}_\omega$ , so  $\rho(\bar{T}(G)) = 1$ , whence  $\rho(N) = 1$ . Therefore  $\rho$  factors via a unique continuous homomorphism  $G/N \rightarrow Z$ .

(b) This follows from (a) and the five term sequence in profinite cohomology [NSW08, Prop. 1.6.7]. □

**3.5. Remarks.** (1) The exact sequence in Lemma 3.4(b) is functorial. Namely, for every continuous homomorphism  $\pi: G \rightarrow G'$  of profinite groups, and every closed normal subgroups  $N$  of  $G$  and  $N'$  of  $G'$  such that  $N \leq \bar{T}(G)$ ,  $N' \leq \bar{T}(G')$ , and  $\pi[N] \leq N'$ , there is a commutative diagram with exact rows

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(N')^{G'} & \xrightarrow{\text{trg}} & H^2(G'/N') & \xrightarrow{\text{inf}} & H^2(G') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(N)^G & \xrightarrow{\text{trg}} & H^2(G/N) & \xrightarrow{\text{inf}} & H^2(G), \end{array}$$

where the vertical maps are induced by  $\pi$ .

(2) In particular, when  $G = G'$  and  $\pi = \text{id}$ , the snake lemma yields an isomorphism between the kernels of the left and middle vertical maps in (3.2).

#### 4. LIFTABLE HOMOMORPHISMS

Let  $G$  be a profinite group, let  $N$  be a closed normal subgroup of  $G$  contained in  $\bar{T}(G)$ , and let  $\pi: G \rightarrow G/N$  be the natural epimorphism. Consider  $\omega \in \Omega$  and a continuous homomorphism  $\bar{\rho}: G/N \rightarrow \bar{\mathbb{U}}_\omega$ . We say that  $\bar{\rho}$  is a  $\pi$ -*liftable* homomorphism if there exists a continuous homomorphism  $\rho: G \rightarrow \mathbb{U}_\omega$  with a commutative square

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ \rho \downarrow & & \downarrow \bar{\rho} \\ \mathbb{U}_\omega & \xrightarrow{\lambda_\omega} & \bar{\mathbb{U}}_\omega. \end{array}$$

We write  $\mathbb{U}_\omega \times_{\bar{\mathbb{U}}_\omega} (G/N)$  for the fiber product with respect to  $\lambda_\omega$  and  $\bar{\rho}$ .

Let  $\alpha_\omega \in H^2(\bar{\mathbb{U}}_\omega)$  be the classifying element of the extension  $\omega$  under the Schreier correspondence [NSW08, Th. 1.2.4]. Let  $\bar{\rho}^*(\alpha_\omega)$  be the pullback of  $\alpha_\omega$  to  $H^2(G/N)$  via  $\bar{\rho}$ . It corresponds to the central extension

$$0 \rightarrow Z \rightarrow \mathbb{U}_\omega \times_{\bar{\mathbb{U}}_\omega} (G/N) \rightarrow G/N \rightarrow 1,$$

where the right map is the projection map [GS06, Remark 3.3.11].



Now consider the following commutative diagram:

$$(4.1) \quad \begin{array}{ccccccc} & & & & G & & \\ & & & & \swarrow \Psi & \searrow \pi & \\ 0 & \longrightarrow & Z & \longrightarrow & \mathbb{U}_\omega \times_{\bar{\mathbb{U}}_\omega} (G/N) & \longrightarrow & G/N \longrightarrow 1 \\ & & \parallel & & \downarrow \rho & & \downarrow \bar{\rho} \\ 0 & \longrightarrow & Z & \longrightarrow & \mathbb{U}_\omega & \xrightarrow{\lambda_\omega} & \bar{\mathbb{U}}_\omega \longrightarrow 1. \end{array}$$

**Proposition 4.1.** *In this set-up, the following conditions are equivalent:*

- (a)  $\bar{\rho}$  is  $\pi$ -liftable, i.e., there exists a continuous homomorphism  $\rho: G \rightarrow \mathbb{U}_\omega$  making the lower triangle commutative;
- (b) There exists a continuous homomorphism  $\Psi: G \rightarrow \mathbb{U}_\omega \times_{\bar{\mathbb{U}}_\omega} (G/N)$  making the upper triangle commutative;
- (c) The inflation of  $\bar{\rho}^*(\alpha_\omega)$  to  $H^2(G)$  is trivial;
- (d) There is a (necessarily unique)  $\psi \in H^1(N)^G$  with  $\bar{\rho}^*(\alpha_\omega) = \text{trg}(\psi)$ .

*Proof.* (a) $\Leftrightarrow$ (b): Use the universal property of the fiber product.

(b) $\Leftrightarrow$ (c): This follows from Hoechsmann's lemma ([Hoe68, 1.1], [NSW08, Prop. 3.5.9]).

(c) $\Leftrightarrow$ (d): Apply Lemma 3.4(b). □

**4.2. Remarks.** (1) In this situation,  $\text{Ker}(\Psi) = \text{Ker}(\rho) \cap N$ .

(2) Since  $N \leq \bar{T}(G)$ , every continuous homomorphism  $\rho: G \rightarrow \mathbb{U}_\omega$ , with  $\omega \in \Omega$ , induces a  $\pi$ -liftable  $\bar{\rho}: G/N \rightarrow \bar{\mathbb{U}}_\omega$ .

(3) If  $\text{cd}_p(G) \leq 1$  for every prime divisor  $p$  of the order of  $Z$ , then the embedding problem in the lower triangle of (4.1) is solvable [NSW08, Th. 3.5.6], i.e.,  $\bar{\rho}$  is  $\pi$ -liftable.

(4) By the explicit description of  $\text{trg}$  [NSW08, Prop. 1.6.6], the homomorphism  $\psi$  in (d) is the restriction  $\Psi|_N$ . Considering it as a map to the left factor of  $\mathbb{U}_\omega \times_{\bar{\mathbb{U}}_\omega} (G/N)$ , we thus have  $\psi = \rho|_N$ .

(5) Every continuous homomorphism  $\rho: G \rightarrow \mathbb{U}_\omega$  as above factors via  $G/(N \cap T(G))$ . Hence  $\bar{\rho}$  is  $\pi$ -liftable if and only if it is  $\pi'$ -liftable, where  $\pi': G/(N \cap T(G)) \rightarrow G/N$  is the natural map. Consequently, by the equivalence (a) $\Leftrightarrow$ (c) in Proposition 4.1,  $\bar{\rho}$  is  $\pi$ -liftable if and only if  $\text{inf}(\bar{\rho}^*(\alpha_\omega)) = 0$  in  $H^2(G/(N \cap T(G)))$ .

**Definition 4.3.** We call an element  $\bar{\rho}^*(\alpha_\omega)$  of  $H^2(G/N)$ , where  $\omega \in \Omega$  and  $\bar{\rho}: G/N \rightarrow \bar{U}_\omega$  is a  $\pi$ -liftable continuous homomorphism, a  $\pi$ -liftable pullback.

The  $\pi$ -liftable part of the cohomology group  $H^2(G/N)$  is its subgroup  $H^2(G/N)_\pi$  generated by all  $\pi$ -liftable pullbacks  $\bar{\rho}^*(\alpha_\omega)$ .

## 5. THE $A$ -PAIRING

From now on we assume that

$$Z \cong \mathbb{Z}/m$$

for an integer  $m \geq 2$ . Thus for a profinite group  $G$  we set  $H^i(G) = H^i(G, \mathbb{Z}/m)$ . For a closed normal subgroup  $N$  of  $G$ , the conjugation induces a  $G$ -action on  $H^1(N)$  [NSW08, Ch. I, §5]. The  $G$ -invariant part of  $H^1(N)$  satisfies

$$H^1(N)^G \cong \text{Hom}_{\text{cont}}(N/N^m[G, N], \mathbb{Z}/m),$$

which is the Pontrjagin dual  $(N/N^m[G, N])^\vee$  of the torsion abelian group  $N/N^m[G, N]$ . Thus the substitution map

$$N/N^m[G, N] \times H^1(N)^G \rightarrow \mathbb{Z}/m$$

is a perfect bilinear map.

For closed normal subgroups  $N_1 \leq N_2$  of  $G$  there is a commutative diagram of perfect bilinear maps

$$\begin{array}{ccc} N_1/N_1^m[G, N_1] & \times & H^1(N_1)^G \longrightarrow \mathbb{Z}/m \\ \downarrow & & \uparrow \text{res} \\ N_2/N_2^m[G, N_2] & \times & H^1(N_2)^G \longrightarrow \mathbb{Z}/m. \end{array}$$

By Lemma 2.1(b), it gives rise to a bilinear map with trivial right kernel

$$(5.1) \quad N_2/N_1N_2^m[G, N_2] \times \text{Ker}(H^1(N_2)^G \xrightarrow{\text{res}} H^1(N_1)) \rightarrow \mathbb{Z}/m.$$

We set

$$A_G(N_1, N_2) = \text{Ker}(H^2(G/N_2) \xrightarrow{\text{inf}} H^2(G/N_1)).$$

By Remark 3.5(2), these two kernels are isomorphic via transgression, so (5.1) induces a bilinear map with trivial right kernel

$$\langle \cdot, \cdot \rangle_{N_1, N_2}^A: N_2/N_1N_2^m[G, N_2] \times A_G(N_1, N_2) \rightarrow \mathbb{Z}/m.$$

Moreover, Corollary 2.2 gives:

**Proposition 5.1.** *When  $m = p$  is prime,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  is perfect.*

**Remark 5.2.** This construction is functorial. Namely, let  $\pi: G \rightarrow G'$  be a continuous homomorphism of profinite groups. Let  $N_1 \leq N_2$  be closed normal subgroups of  $G$ , let  $N'_1 \leq N'_2$  be closed normal subgroups of  $G'$ , and suppose that  $\pi[N_1] \leq N'_1$  and  $\pi[N_2] \leq N'_2$ . Then  $\pi$  induces a commutative diagram

$$\begin{array}{ccc} N_2/N_1 N_2^m[G, N_2] & \times & A_G(N_1, N_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^A} \mathbb{Z}/m \\ \downarrow & & \uparrow \\ N'_2/N'_1 (N'_2)^m[G', N'_2] & \times & A_{G'}(N'_1, N'_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N'_1, N'_2}^A} \mathbb{Z}/m. \end{array}$$

## 6. THE $B$ -PAIRING

Let  $G$  be a profinite group, and let  $N_1 \leq N_2$  be closed normal subgroups of  $G$  contained in  $\bar{T}(G)$ . Let  $\pi_i: G \rightarrow G/N_i$ ,  $i = 1, 2$ , be the natural projections. Every  $\pi_1$ -liftable continuous homomorphism  $\bar{\rho}: G/N_1 \rightarrow \bar{U}_\omega$ ,  $\omega \in \Omega$ , factors via a  $\pi_2$ -liftable continuous homomorphism  $\bar{\rho}: G/N_2 \rightarrow \bar{U}_\omega$ . Hence the inflation homomorphism maps  $H^2(G/N_2)_{\pi_2}$  onto  $H^2(G/N_1)_{\pi_1}$  (see Definition 4.3). We set

$$B_G(N_1, N_2) = \left\langle \bar{\rho}^*(\alpha_\omega) \in A_G(N_1, N_2) \mid \omega \in \Omega, \bar{\rho}: G/N_2 \rightarrow \bar{U}_\omega \text{ } \pi_2\text{-liftable} \right\rangle.$$

Thus  $B_G(N_1, N_2)$  is generated by all  $\pi_2$ -liftable pullbacks in  $H^2(G/N_2)_{\pi_2}$  which vanish under the inflation to  $H^2(G/N_1)$ .

**Theorem 6.1.** *In this setup, there is a non-degenerate bilinear map*

$$\langle \cdot, \cdot \rangle_{N_1, N_2}^B: N_2/(N_2 \cap \pi_1^{-1}[T(G/N_1)]) \times B_G(N_1, N_2) \rightarrow \mathbb{Z}/m$$

which fits into a commutative diagram

$$\begin{array}{ccc} N_2/N_1 N_2^m[G, N_2] & \times & A_G(N_1, N_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^A} \mathbb{Z}/m \\ \downarrow & & \uparrow \\ N_2/(N_2 \cap \pi_1^{-1}[T(G/N_1)]) & \times & B_G(N_1, N_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^B} \mathbb{Z}/m. \end{array}$$

Moreover, when  $m = p$  is prime,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  is perfect.

*Proof.* First we note that, by Lemma 3.3 and (3.1),

$$(6.1) \quad \begin{aligned} N_1 N_2^m[G, N_2] &\leq N_2 \cap N_1 \bar{T}(G)^m[G, \bar{T}(G)] \leq N_2 \cap N_1 T(G) \\ &\leq N_2 \cap \pi_1^{-1}[T(G/N_1)]. \end{aligned}$$

Now let  $B_0$  be the above set of generators of  $B_G(N_1, N_2)$ , i.e.,  $B_0$  is the set of all  $\pi_2$ -liftable pullbacks  $\bar{\rho}^*(\alpha_\omega)$  such that  $\inf_{G/N_1}(\bar{\rho}^*(\alpha_\omega)) = 0$ .

By Proposition 4.1, the latter condition means that  $\bar{\rho}^*(\alpha_\omega)$  is in fact  $\pi_{12}$ -liftable, where  $\pi_{12}: G/N_1 \rightarrow G/N_2$  is the natural projection. Equivalently, there is a (necessarily unique)  $\psi \in H^1(N_2/N_1)^{G/N_1}$  such that  $\bar{\rho}^*(\alpha_\omega) = \text{trg}(\psi)$ , for the transgression map  $\text{trg}: H^1(N_2/N_1)^{G/N_1} \rightarrow H^2(G/N_2)$ . For a  $\pi_{12}$ -lifting  $\rho$  of  $\bar{\rho}$ , we have  $\rho|_{N_2/N_1} = \psi$  (Remark 4.2(4)).

Next let  $\sigma \in N_2$ , and let  $\bar{\sigma}$  be its coset modulo  $N_1N_2^m[G, N_2]$ . Then  $\bar{\sigma}$  is in the annihilator of  $B_G(N_1, N_2)$  with respect to  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  if and only if it is annihilated by all generators in  $B_0$ . This means that for every  $\bar{\rho}, \psi, \rho$  as in the previous paragraph,

$$0 = \langle \bar{\sigma}, \bar{\rho}^*(\alpha_\omega) \rangle_{N_1, N_2}^A = \langle \bar{\sigma}, \text{trg}(\psi) \rangle_{N_1, N_2}^A = \psi(\sigma N_1) = \rho(\sigma N_1).$$

Equivalently,  $\sigma \in N_2 \cap \pi_1^{-1}[T(G/N_1)]$ .

Consequently,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  induces a bilinear map  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  as in the assertion, whose left kernel is trivial. In light of (6.1), this bilinear map fits into a commutative diagram as claimed. Since the right kernel of  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  is trivial and the left (resp., right) vertical map of the diagram is surjective (resp., injective), the right kernel of  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  is also trivial, i.e.,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  is non-degenerate.

When  $m = p$  is prime,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  is perfect, whence left-surjective. It follows from Lemma 2.3 that  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  is also left-surjective. Since it is non-degenerate, Pontrjagin duality implies that it is perfect.  $\square$

**Corollary 6.2.** *In the setup of Theorem 6.1, and for  $m = p$  prime, the following conditions are equivalent:*

- (a)  $N_1N_2^p[G, N_2] = N_2 \cap \pi_1^{-1}[T(G/N_1)]$ ;
- (b) *There is an exact sequence*

$$0 \rightarrow B_G(N_1, N_2) \hookrightarrow H^2(G/N_2) \xrightarrow{\text{inf}} H^2(G/N_1).$$

**Remark 6.3.** The bilinear map  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  is functorial in the following sense:

Let  $\pi: G \rightarrow G'$  be a continuous map of profinite groups. Let  $N_1 \leq N_2$  be closed normal subgroups of  $G$  contained in  $\bar{T}(G)$ , let  $N'_1 \leq N'_2$  be closed normal subgroups of  $G'$  contained in  $\bar{T}(G')$ , and suppose that  $\pi[N_1] \leq N'_1$  and  $\pi[N_2] \leq N'_2$ . We write  $\pi_i: G \rightarrow G/N_i$ ,  $\pi'_i: G' \rightarrow G'/N'_i$ ,  $i = 1, 2$ , for the natural projections. Then the functoriality of  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$  and  $\langle \cdot, \cdot \rangle_{N'_1, N'_2}^A$  (Remark 5.2) and the construction of the pairing in Theorem 6.1 give rise

to a commutative diagram of non-degenerate bilinear maps

$$\begin{array}{ccc}
 N_2/(N_2 \cap \pi_1^{-1}[T(G/N_1)]) & \times & B_G(N_1, N_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^B} \mathbb{Z}/m \\
 \downarrow & & \uparrow \\
 N'_2/(N'_2 \cap (\pi'_1)^{-1}[T(G'/N'_1)]) & \times & B_{G'}(N'_1, N'_2) \xrightarrow{\langle \cdot, \cdot \rangle_{N'_1, N'_2}^B} \mathbb{Z}/m.
 \end{array}$$

As a special case we obtain:

**Proposition 6.4.** *Let  $N$  be a closed normal subgroup of  $G$  contained in  $\bar{T}(G)$ , and let  $\pi: G \rightarrow G/N$  be the natural projection.*

(a) *There is a non-degenerate bilinear map*

$$N/(N \cap T(G)) \times H^2(G/N)_\pi \rightarrow \mathbb{Z}/m.$$

(b) *When  $m = p$  is prime, there is an exact sequence*

$$0 \rightarrow H^2(G/N)_\pi \rightarrow H^2(G/N) \xrightarrow{\text{inf}} H^2(G/(N \cap T(G))).$$

*Proof.* Take in Theorem 6.1  $N_1 = N \cap T(G)$  and  $N_2 = N$ . By Remark 4.2(5),  $\text{inf}: H^2(G/N_2)_\pi \rightarrow H^2(G/N_1)$  is trivial, so  $B_G(N_1, N_2) = H^2(G/N)_\pi$ . Further, by Lemma 3.2,  $\pi_1^{-1}[T(G/N_1)] = T(G)$ , and we deduce (a).

By Lemma 3.3,  $N^m[G, N] \leq N \cap T(G) = N_1$ , so  $N_1 N^m[G, N] = N \cap T(G)$ . Therefore Corollary 6.2 gives (b).  $\square$

## 7. THE $C$ -PAIRING

Let  $N_1 \leq N_2$  be again closed normal subgroups of the profinite group  $G$  which are contained in  $\bar{T}(G)$ . Let  $\pi_i: G \rightarrow G/N_i$ ,  $i = 1, 2$ , be the canonical projections. We set

$$C_G(N_1, N_2) = \text{Ker}(\text{inf}: H^2(G/N_2)_{\pi_2} \rightarrow H^2(G/N_1)_{\pi_1}).$$

Then

$$B_G(N_1, N_2) \leq C_G(N_1, N_2) \leq A_G(N_1, N_2).$$

**Theorem 7.1.** *Suppose that  $m = p$  is prime. Then  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  induces a perfect bilinear map*

$$\langle \cdot, \cdot \rangle_{N_1, N_2}^C: N_2/(N_2 \cap N_1 T(G)) \times C_G(N_1, N_2) \rightarrow \mathbb{Z}/p.$$

*Proof.* Let

$$\tilde{\pi}_1: G \rightarrow G/(N_1 \cap T(G)), \quad \tilde{\pi}_2: G \rightarrow G/(N_2 \cap T(G))$$

be the canonical projections. The functoriality of  $\langle \cdot, \cdot \rangle^B$  (Remark 6.3) gives rise to a commutative diagram of perfect bilinear maps

$$\begin{array}{ccc} N_1/(N_1 \cap \tilde{\pi}_1^{-1}[T(G/(N_1 \cap T(G)))] & \times & B_G(N_1 \cap T(G), N_1) \xrightarrow{\langle \cdot, \cdot \rangle^B} \mathbb{Z}/p \\ \downarrow & & \uparrow \parallel \\ N_2/(N_2 \cap \tilde{\pi}_2^{-1}[T(G/(N_2 \cap T(G)))] & \times & B_G(N_2 \cap T(G), N_2) \xrightarrow{\langle \cdot, \cdot \rangle^B} \mathbb{Z}/p, \end{array}$$

(where we omit the obvious subscripts on  $\langle \cdot, \cdot \rangle^B$ ). By Lemma 3.2,

$$\tilde{\pi}_1^{-1}[T(G/(N_1 \cap T(G)))] = T(G), \quad \tilde{\pi}_2^{-1}[T(G/(N_2 \cap T(G)))] = T(G).$$

In view of Remark 4.2(5),

$$B_G(N_1 \cap T(G), N_1) = H^2(G/N_1)_{\pi_1}, \quad B_G(N_2 \cap T(G), N_2) = H^2(G/N_2)_{\pi_2}.$$

We obtain a commutative diagram of perfect bilinear maps

$$\begin{array}{ccc} N_1/(N_1 \cap T(G)) & \times & H^2(G/N_1)_{\pi_1} \xrightarrow{\langle \cdot, \cdot \rangle^B} \mathbb{Z}/p \\ \downarrow & & \uparrow \parallel \\ N_2/(N_2 \cap T(G)) & \times & H^2(G/N_2)_{\pi_2} \xrightarrow{\langle \cdot, \cdot \rangle^B} \mathbb{Z}/p. \end{array}$$

By Corollary 2.2, it induces a perfect bilinear map between the cokernel  $N_2/N_1(N_2 \cap T(G)) \cong N_2/(N_2 \cap N_1 T(G))$  of the left vertical map and the kernel  $C_G(N_1, N_2)$  of the right vertical map, as required.  $\square$

**Remark 7.2.** The bilinear map  $\langle \cdot, \cdot \rangle_{N_1, N_2}^C$  is functorial in a similar manner to  $\langle \cdot, \cdot \rangle_{N_1, N_2}^A$ ,  $\langle \cdot, \cdot \rangle_{N_1, N_2}^B$  (see Remarks 5.2 and 6.3).

**Corollary 7.3.** *The following conditions are equivalent:*

- (a)  $\text{inf}: H^2(G/N_2)_{\pi_2} \rightarrow H^2(G/N_1)_{\pi_1}$  is an isomorphism;
- (b)  $\text{inf}: H^2(G/N_2)_{\pi_2} \rightarrow H^2(G/N_1)_{\pi_1}$  is a monomorphism;
- (c)  $N_1 T(G) = N_2 T(G)$ .

*Proof.* We have noticed in §6 that  $\text{inf}$  is surjective, so (a) and (b) are equivalent. The equivalence of (b) and (c) follows from Theorem 7.1.  $\square$

We summarize Theorem 6.1 (for  $m = p$  prime) and Theorem 7.1 by the following commutative diagram of perfect bilinear maps. Here  $N_1 \leq N_2$

are closed normal subgroups of  $G$  contained in  $\bar{T}(G)$ :

$$(7.1) \quad \begin{array}{ccccc} N_2/N_1N_2^p[G, N_2] & \times & A_G(N_1, N_2) & \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^A} & \mathbb{Z}/p \\ \downarrow & & \uparrow & & \parallel \\ N_2/(N_2 \cap N_1T(G)) & \times & C_G(N_1, N_2) & \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^C} & \mathbb{Z}/p \\ \downarrow & & \uparrow & & \parallel \\ N_2/(N_2 \cap \pi_1^{-1}[T(G/N_1)]) & \times & B_G(N_1, N_2) & \xrightarrow{\langle \cdot, \cdot \rangle_{N_1, N_2}^B} & \mathbb{Z}/p. \end{array}$$

**Definition 7.4.** We say that the *kernel generating condition* holds for the subgroups  $N_1, N_2$  of  $G$  if  $B_G(N_1, N_2) = C_G(N_1, N_2)$ .

We now arrive at the main result of the construction, which characterizes the transfer condition  $\pi[T(G)] = T(\pi[G])$  in cohomological terms:

**Theorem 7.5.** *Let  $m = p$  be prime. Let  $N$  be a closed normal subgroup of the profinite group  $G$  contained in  $\bar{T}(G)$ , and let  $\pi: G \rightarrow G/N$  be the natural projection. The following conditions are equivalent:*

- (a)  $\pi[T(G)] = T(G/N)$ ;
- (b) *The kernel generating condition holds for the subgroups  $\bar{T}(G), N$  of  $G$ .*

**Remark 7.6.** Condition (b) can be stated more explicitly as follows: Let  $\bar{\pi}: G \rightarrow G/\bar{T}(G)$  be the natural projection, and consider the inflation map  $\text{inf}: H^2(G/\bar{T}(G)) \rightarrow H^2(G/N)$ . Let  $\Pi$  be the set of all pullbacks  $\bar{\rho}^*(\alpha_\omega)$ , with  $\omega \in \Omega$  and  $\bar{\rho}: G/\bar{T}(G) \rightarrow \bar{U}_\omega$  is a  $\bar{\pi}$ -liftable continuous homomorphism. Then the subgroups  $\langle \Pi \rangle, \langle \Pi \cap \text{Ker}(\text{inf}) \rangle$  of  $H^2(G/\bar{T}(G))$  generated by  $\Pi, \Pi \cap \text{Ker}(\text{inf})$ , respectively, satisfy

$$\langle \Pi \cap \text{Ker}(\text{inf}) \rangle = \langle \Pi \rangle \cap \text{Ker}(\text{inf}).$$

*Proof.* First note that (a) is equivalent to  $NT(G) = \pi^{-1}[T(G/N)]$ . As  $N \leq \bar{T}(G)$ , Lemma 3.2 implies that

$$\pi^{-1}[T(G/N)] \leq \pi^{-1}[\bar{T}(G/N)] = \bar{T}(G).$$

From (7.1) we therefore obtain a commutative diagram of perfect bilinear maps

$$\begin{array}{ccccc} \bar{T}(G)/NT(G) & \times & C_G(N, \bar{T}(G)) & \xrightarrow{\langle \cdot, \cdot \rangle_{N, \bar{T}(G)}^C} & \mathbb{Z}/p \\ \downarrow & & \uparrow & & \parallel \\ \bar{T}(G)/\pi^{-1}[T(G/N)] & \times & B_G(N, \bar{T}(G)) & \xrightarrow{\langle \cdot, \cdot \rangle_{N, \bar{T}(G)}^B} & \mathbb{Z}/p. \end{array}$$

Hence the left vertical map is an equality if and only if the right vertical map is an equality. This gives the required equivalence.  $\square$

## 8. UNIPOTENT UPPER-TRIANGULAR MATRICES

We fix an integer  $n \geq 2$ . For a commutative unital profinite ring  $R$  let  $\mathbb{U}_n(R)$  be the group of all unipotent upper-triangular  $(n+1) \times (n+1)$ -matrices over  $R$ . The additive group  $R^+$  of  $R$  embeds as a central subgroup of  $\mathbb{U}_n(R)$  via  $r \mapsto I_{n+1} + rE_{1,n+1}$ , where  $I_{n+1}$  is the identity matrix and  $E_{1,n+1}$  is the matrix with 1 at entry  $(1, n+1)$  and 0 elsewhere. Setting  $\bar{\mathbb{U}}_n(R) = \mathbb{U}_n(R)/R^+$  we obtain a central extension

$$\omega: \quad 0 \rightarrow R^+ \rightarrow \mathbb{U}_n(R) \rightarrow \bar{\mathbb{U}}_n(R) \rightarrow 1.$$

We define maps  $\gamma_1, \gamma_2: \bar{\mathbb{U}}_n(R) \rightarrow \mathbb{U}_n(R)$  by mapping a coset  $\bar{M}$  of  $M \in \mathbb{U}_n(R)$  to the matrix in  $\mathbb{U}_n(R)$  which is 0 at entries  $(1, j)$ ,  $j = 2, 3, \dots, n+1$  (resp.,  $(i, n+1)$ ,  $i = 1, 2, \dots, n$ ), and which coincides with  $M$  elsewhere. It is straightforward to verify that  $\gamma_1, \gamma_2$  are homomorphisms, and  $\text{Ker}(\gamma_1) \cap \text{Ker}(\gamma_2) = \{I_{n+1}\}$ . Therefore assumption (I) of §3 is satisfied for  $\omega$ . Assumption (II) is satisfied when  $p||R| < \infty$ .

**Lemma 8.1.** *For every profinite group  $G$  one has  $T^{\bar{\mathbb{U}}_n(R)}(G) = T^{\mathbb{U}_{n-1}(R)}(G)$ .*

*Proof.* There is an embedding  $\mathbb{U}_{n-1}(R) \hookrightarrow \bar{\mathbb{U}}_n(R)$  into the upper-left  $n \times n$ -block, and with zeros at entries  $(i, n+1)$ ,  $i = 2, \dots, n$ . By Lemma 3.1, it implies that  $T^{\bar{\mathbb{U}}_n(R)}(G) \leq T^{\mathbb{U}_{n-1}(R)}(G)$ . Moreover, applying Lemma 3.1 for the projections  $\bar{\mathbb{U}}_n(R) \rightarrow \mathbb{U}_{n-1}(R)$  on the upper-left and lower-right  $n \times n$  blocks shows that  $T^{\mathbb{U}_{n-1}(R)}(G) \leq T^{\bar{\mathbb{U}}_n(R)}(G)$ .  $\square$

Now suppose that the ring  $R$  is finite. Let  $\bar{G}$  be a profinite group. Given a continuous homomorphism  $\bar{\rho}: \bar{G} \rightarrow \bar{\mathbb{U}}_n(R)$ , we write  $\bar{\rho}_{ij}$  for the  $(i, j)$ -entry of  $\bar{\rho}$ . The maps  $\bar{\rho}_{i,i+1}: \bar{G} \rightarrow R^+$  are group homomorphisms. The pullbacks  $\bar{\rho}^*(\alpha_\omega)$  are the elements of the  $n$ -fold Massey products  $\langle \varphi_1, \dots, \varphi_n \rangle \subseteq H^2(\bar{G}, R^+)$ , where  $\varphi_1, \dots, \varphi_n \in H^1(\bar{G}, R^+) = \text{Hom}_{\text{cont}}(\bar{G}, R^+)$  [Dwy75, Th. 2.4]. Specifically,  $\langle \varphi_1, \dots, \varphi_n \rangle$  consists of all pullbacks  $\bar{\rho}^*(\alpha_\omega)$  such that  $\varphi_i = \bar{\rho}_{i,i+1}$ ,  $i = 1, 2, \dots, n$ . For our purposes, this can be taken as the definition of the Massey product in the profinite group setting.

In particular, when  $n = 2$  one has  $\langle \varphi_1, \varphi_2 \rangle = \{\varphi_1 \cup \varphi_2\}$ , where the cup product is induced from the product map  $R \otimes R \rightarrow R$ . Hence  $\bar{\rho}^*(\alpha_\omega) = \bar{\rho}_{12} \cup \bar{\rho}_{23}$  [EfMi11, Prop. 9.1]. Since any  $\varphi_1, \varphi_2 \in H^1(\bar{G}, R^+)$  can be realized as  $\bar{\rho}_{12}, \bar{\rho}_{23}$ , respectively, for some  $\bar{\rho}$ , we obtain in this way all cup products  $\varphi_1 \cup \varphi_2$  in  $H^2(\bar{G}, R^+)$ .



9. THE  $p$ -ZASSENHAUS FILTRATION

We now focus on the ring  $R = \mathbb{Z}/p$ , with  $p$  prime. We abbreviate

$$\mathbb{U}_n = \mathbb{U}_n(\mathbb{Z}/p), \quad \bar{\mathbb{U}}_n = \bar{\mathbb{U}}_n(\mathbb{Z}/p).$$

Let  $\Omega = \{\omega\}$ , where  $\omega = \omega_n$  is the central extension

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{U}_n \rightarrow \bar{\mathbb{U}}_n \rightarrow 1.$$

For a profinite group  $G$  we have, by Lemma 8.1,

$$T(G) = T^{\mathbb{U}_n}(G), \quad \bar{T}(G) = T^{\bar{\mathbb{U}}_n}(G) = T^{\mathbb{U}_{n-1}}(G).$$

We set  $\bar{G} = G/\bar{T}(G) = G/T^{\mathbb{U}_{n-1}}(G)$  and let  $\pi: G \rightarrow \bar{G}$  be the natural projection. In view of the remarks in §8, we denote the subgroup of  $H^2(\bar{G})$  generated by all  $\bar{\rho}^*(\alpha_\omega)$ , for continuous homomorphisms  $\bar{\rho}: \bar{G} \rightarrow \bar{\mathbb{U}}_n$ , by  $H^2(\bar{G})_{n\text{-Massey}}$ . Recall that  $H^2(\bar{G})_\pi$  is the subgroup of  $H^2(\bar{G})$  generated by all such pullbacks which become trivial under the inflation to  $H^2(G)$ . We denote it in the current setting by  $H^2(\bar{G})_{n\text{-Massey}, G}$ .

Now Proposition 6.4 gives:

**Theorem 9.1.** (a) *The bilinear map  $\langle \cdot, \cdot \rangle^B$  gives a perfect bilinear map*

$$T^{\mathbb{U}_{n-1}}(G)/T^{\mathbb{U}_n}(G) \times H^2(G/T^{\mathbb{U}_{n-1}}(G))_{n\text{-Massey}, G} \rightarrow \mathbb{Z}/p,$$

(b) *There is an exact sequence*

$$0 \rightarrow H^2(G/T^{\mathbb{U}_{n-1}}(G))_{n\text{-Massey}, G} \hookrightarrow H^2(G/T^{\mathbb{U}_{n-1}}(G)) \xrightarrow{\text{inf}} H^2(G/T^{\mathbb{U}_n}(G)).$$

Corollary 7.3 gives in this setting:

**Proposition 9.2.** *Let  $N_1 \leq N_2$  be closed normal subgroups of  $G$  contained in  $T^{\mathbb{U}_{n-1}}(G)$ . The following conditions are equivalent:*

- (a)  $H^2(G/N_i)_{n\text{-Massey}, G}$ ,  $i = 1, 2$ , are isomorphic via inflation;
- (b)  $N_1 T^{\mathbb{U}_n}(G) = N_2 T^{\mathbb{U}_n}(G)$ .

The above structural results can sometimes be rephrased in terms of the  $p$ -Zassenhaus filtration  $G_{(n,p)}$ ,  $n = 1, 2, 3, \dots$ , of  $G$  (see the Introduction). By its definition, the quotients  $G_{(n-1,p)}/G_{(n,p)}$  are elementary abelian  $p$ -groups, so by induction,  $G/G_{(n,p)}$  is a pro- $p$  group for every  $n$ .

When  $S$  is a free profinite group it was shown in [Efr14a, Th. A] that

$$(9.1) \quad T(S) = T^{\mathbb{U}_n}(S) = S_{(n+1,p)}.$$

Consequently,  $\bar{T}(S) = T^{\mathbb{U}_{n-1}}(S) = S_{(n,p)}$ . See also [MT15, Th. 2.7] and [Efr14b] for alternative proofs.

Therefore Theorem 9.1 and Proposition 9.2, when applied to  $G = S$ , recover [Efr14a, Th. B] and [Efr14a, Cor. 10.2], respectively.

Now Theorem 7.5 can be interpreted in this setting as follows:

**Theorem 9.3.** *Let  $S$  be a free profinite group, let  $N$  be a closed normal subgroup of  $S$  contained in  $S_{(n,p)}$ , and let  $G = S/N$ . Then:*

- (a)  $T^{\mathbb{U}_{n-1}}(G) = G_{(n,p)}$ .
- (b) One has  $T^{\mathbb{U}_n}(G) = G_{(n+1,p)}$  if and only if

$$\text{Ker}(H^2(G/G_{(n,p)})_{n\text{-Massey}} \xrightarrow{\text{inf}} H^2(G))$$

*is generated by elements of  $n$ -fold Massey products.*

*Proof.* Let  $\pi: S \rightarrow G$  be the natural projection.

- (a) As  $N \leq \bar{T}(S)$ , Lemma 3.2 gives:

$$T^{\mathbb{U}_{n-1}}(G) = \bar{T}(G) = \pi[\bar{T}(S)] = \pi[T^{\mathbb{U}_{n-1}}(S)] = \pi[S_{(n,p)}] = G_{(n,p)}.$$

(b) One has  $S/S_{(n,p)} \cong G/G_{(n,p)}$  via  $\pi$ . Hence the condition about the kernel in (b) means that  $\text{Ker}(H^2(S/S_{(n,p)})_{n\text{-Massey}} \rightarrow H^2(S/N))$  is generated by  $n$ -fold Massey products. This is exactly the kernel generating condition for the subgroups  $N, S_{(n,p)}$  of  $S$  (see Remark 4.2(3)). By Theorem 7.5 (with  $G$  replaced by  $S$ ), it is equivalent to  $T(G) = \pi[T(S)]$ , i.e., to  $T^{\mathbb{U}_n}(G) = G_{(n+1,p)}$ .  $\square$

The condition about the kernel in Theorem 9.3(b) is called in [Efr14a] the  *$n$ -Massey kernel condition* on  $G$ . Thus the ‘‘if’’ part in the equivalence recovers [Efr14a, Th. A’].

**Remark 9.4.** Denote the maximal pro- $p$  quotient of  $G$  by  $G(p)$ . Let  $\lambda: G \rightarrow G(p)$  be the natural epimorphism. Then  $\text{Ker}(\lambda)$  is contained in both  $T(G) = T^{\mathbb{U}_n(\mathbb{Z}/p)}(G)$  and  $G_{(n+1,p)}$ . By Lemma 3.2,  $\lambda[T(G)] = T(G(p))$ . Also,  $\lambda[G_{(n+1,p)}] = G(p)_{(n+1,p)}$ . Consequently,  $T(G) = G_{(n+1,p)}$  if and only if  $T(G(p)) = (G(p))_{(n+1,p)}$  (and similarly for  $\bar{T}(G), G_{(n,p)}$ ).

## 10. THE LOWER $p$ -CENTRAL FILTRATION

We fix  $n \geq 2$ . For every  $1 \leq s \leq n$  let  $\mathbb{U}_{n,s} = \mathbb{U}_s(\mathbb{Z}/p^{n-s+1})$  and  $\bar{\mathbb{U}}_{n,s} = \bar{\mathbb{U}}_{n,s}(\mathbb{Z}/p^{n-s+1})$  (with notation as in §8). Let  $\Omega$  consist of the central extensions

$$\omega_{n,s}: \quad 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{U}_{n,s} \rightarrow \bar{\mathbb{U}}_{n,s} \rightarrow 1, \quad s = 1, 2, \dots, n.$$

For example, the extensions  $\omega_{n,1}, \omega_{n,n}$  are

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^{n-1} \rightarrow 0, \quad 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{U}_n(\mathbb{Z}/p) \rightarrow \bar{\mathbb{U}}_n(\mathbb{Z}/p) \rightarrow 1,$$

respectively. As noted in §8, assumptions (I) and (II) are satisfied.

For every profinite group  $G$  we clearly have

$$T(G) = \bigcap_{s=1}^n T^{\mathbb{U}_{n,s}}(G).$$

**Proposition 10.1.** *One has  $\bar{T}(G) = \bigcap_{s=1}^{n-1} T^{\mathbb{U}_{n-1,s}}(G)$ .*

*Proof.* For every  $2 \leq s \leq n$ , the group  $\mathbb{U}_{n-1,s-1}$  embeds as the upper-left  $s \times s$ -block of  $\bar{\mathbb{U}}_{n,s}$ , with zeros at entries  $(i, s+1)$ ,  $i = 2, 3, \dots, s$ . By Lemma 3.1,

$$T^{\bar{\mathbb{U}}_{n,s}}(G) \leq T^{\mathbb{U}_{n-1,s-1}}(G), \quad s = 2, \dots, n.$$

Moreover,  $\bar{\mathbb{U}}_{n,1} \cong \mathbb{Z}/p^{n-1} \cong \mathbb{U}_{n-1,1}$ , so

$$T^{\bar{\mathbb{U}}_{n,1}}(G) = T^{\mathbb{U}_{n-1,1}}(G).$$

Hence

$$\bar{T}(G) = \bigcap_{s=1}^n T^{\bar{\mathbb{U}}_{n,s}}(G) \leq \bigcap_{s=1}^{n-1} T^{\mathbb{U}_{n-1,s}}(G).$$

On the other hand, for  $2 \leq s \leq n-1$  let  $\gamma_1, \gamma_2: \bar{\mathbb{U}}_{n,s} \rightarrow \mathbb{U}_{n-1,s-1}$  be the projections on the upper-left and lower-right  $s \times s$ -blocks. Also let  $\gamma_3: \bar{\mathbb{U}}_{n,s} \rightarrow \mathbb{U}_{n-1,s}$  be the homomorphism induced by the ring epimorphism  $\mathbb{Z}/p^{n-s+1} \rightarrow \mathbb{Z}/p^{n-s}$ . Then  $\text{Ker}(\gamma_1) \cap \text{Ker}(\gamma_2) \cap \text{Ker}(\gamma_3) = \{I_s\}$ . Lemma 3.1 therefore implies that

$$T^{\mathbb{U}_{n-1,s-1}}(G) \cap T^{\mathbb{U}_{n-1,s}}(G) \leq T^{\bar{\mathbb{U}}_{n,s}}(G), \quad s = 2, \dots, n-1.$$

Moreover, Lemma 8.1 gives  $T^{\mathbb{U}_{n-1,n-1}}(G) = T^{\bar{\mathbb{U}}_{n,n}}(G)$ . We deduce that

$$\bigcap_{s=1}^{n-1} T^{\mathbb{U}_{n-1,s}}(G) \leq \bigcap_{s=1}^n T^{\bar{\mathbb{U}}_{n,s}}(G) = \bar{T}(G),$$

and the assertion follows.  $\square$

When  $S$  is a free profinite group one has

$$T(S) = S^{(n+1,p)}.$$

Indeed, this is proved in [MT15, Th. 2.7] when  $S$  is finitely generated; The general case follows by an inverse limit argument. In view of Proposition 10.1, this implies that  $\bar{T}(S) = S^{(n,p)}$ . See [Efr14b] for a variant of this result.

Given a profinite group  $G$ , we call the pullbacks  $\bar{\rho}^*(\alpha_{\omega_{n,s}})$ , where  $1 \leq s \leq n$  and  $\bar{\rho}: G/\bar{T}(G) \rightarrow \bar{\mathbb{U}}_{n,s}$  is a continuous homomorphism,  $n$ -unitriangular elements. They were studied in [Efr17] under the name *the unitriangular*

spectrum. Let  $H^2(G/\bar{T}(G))_{n\text{-ut}}$  be the subgroup of  $H^2(G/\bar{T}(G))$  generated by all  $n$ -unitriangular, and let  $H^2(G/\bar{T}(G))_{n\text{-ut},G}$  be the subgroup of  $H^2(G/\bar{T}(G))$  generated by all these pullbacks with  $\bar{\rho}$  liftable to  $G$ .

Let  $G^{(n,p)}$ ,  $n = 1, 2, 3, \dots$ , be the lower  $p$ -central filtration of the profinite group  $G$  (see the Introduction). The quotients  $G^{(n,p)}/G^{(n+1,p)}$  are elementary  $p$ -groups, so by induction,  $G/G^{(n,p)}$  is a pro- $p$  group for every  $n$ .

**Proposition 10.2.** *For a free profinite group  $S$  one has*

$$H^2(S/\bar{T}(S))_{n\text{-ut},S} = H^2(S/\bar{T}(S))_{n\text{-ut}} = H^2(S/S^{(n,p)}).$$

*Proof.* The left equality follows from Remark 4.2(3).

The main result of [Efr17] gives a canonical linear basis (called the *Lyndon basis*) for  $H^2(S/S^{(n,p)})$  which is made of certain  $n$ -unitriangular elements  $\bar{\rho}^*(\alpha_{\omega_{n,s}})$ . This gives the right equality.  $\square$

We deduce the analog of Theorem 9.3:

**Theorem 10.3.** *Let  $S$  be a free profinite group, let  $N$  be a closed normal subgroup of  $S$  contained in  $S^{(n,p)}$ , and let  $G = S/N$ . Then:*

- (a)  $\bar{T}(G) = G^{(n,p)}$ .
- (b) *One has  $T(G) = G^{(n+1,p)}$  if and only if*

$$\text{Ker}(H^2(G/G^{(n,p)}) \xrightarrow{\text{inf}} H^2(G))$$

*is generated by  $n$ -unitriangular elements.*

*Proof.* Let  $\pi: S \rightarrow G$  be the natural projection.

- (a) As  $N \leq \bar{T}(S)$ , Lemma 3.2 gives:

$$\bar{T}(G) = \pi[\bar{T}(S)] = \pi[S^{(n,p)}] = G^{(n,p)}.$$

(b) We have  $S/S^{(n,p)} \cong G/G^{(n,p)}$  via  $\pi$ . Hence the condition about the kernel means that  $\text{Ker}(\text{inf}: H^2(S/S^{(n,p)}) \rightarrow H^2(S/N))$  is generated by  $n$ -triangular elements. In view of Proposition 10.2, this is exactly the kernel generating condition for the subgroups  $N, S^{(n,p)}$  of  $S$ , and  $\Omega$  as above. By Theorem 7.5 (with  $G$  replaced by  $S$ ), it is equivalent to  $T(G) = \pi[T(S)]$ , i.e., to  $T(G) = \pi[S^{(n+1,p)}] = G^{(n+1,p)}$ .  $\square$

When the condition on the kernel in (b) holds, we say that  $G$  satisfies the  *$n$ -unitriangular kernel condition*.

From Theorem 10.3 and Proposition 6.4 we deduce:

**Corollary 10.4.** *Let  $S$  be a free profinite group, let  $N$  be a closed normal subgroup of  $S$  contained in  $S^{(n,p)}$ , and suppose that  $G = S/N$  satisfies*

the  $n$ -unitriangular kernel condition. Let  $\pi: G \rightarrow G/G^{(n,p)}$  be the natural projection. Then:

(a) There is a non-degenerate bilinear map

$$G^{(n,p)}/G^{(n+1,p)} \times H^2(G/G^{(n,p)})_\pi \rightarrow \mathbb{Z}/p.$$

(b) There is an exact sequence

$$0 \rightarrow H^2(G/G^{(n,p)})_\pi \rightarrow H^2(G/G^{(n,p)}) \xrightarrow{\text{inf}} H^2(G/G^{(n+1,p)}).$$

**Remark 10.5.** Similarly to Remark 9.4, we have  $T(G) = G^{(n+1,p)}$  if and only  $T(G(p)) = (G(p))^{(n+1,p)}$ , and likewise for  $\bar{T}(G)$ ,  $G^{(n,p)}$ .

### 11. THE LOWER $p$ -CENTRAL FILTRATION FOR $p > 2$ , $n = 2$

The discussion in this section is inspired by [EfMi11]. Let  $p > 2$  and let  $M_{p^3}$  be the extra-special group of order  $p^3$  and exponent  $p^2$ . Thus

$$(11.1) \quad M_{p^3} = \langle r, s \mid r^{p^2} = s^p = 1, [r, s] = r^p \rangle.$$

Let  $\Omega$  consist of the central extensions

$$\omega: 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0, \quad \omega': 0 \rightarrow \mathbb{Z}/p \rightarrow M_{p^3} \rightarrow (\mathbb{Z}/p)^2 \rightarrow 0.$$

where the epimorphism in  $\omega'$  is given by  $r \mapsto (1, 0)$ ,  $s \mapsto (0, 1)$ . The group  $\mathbb{Z}/p$  clearly embeds in  $\mathbb{Z}/p^2$ , and the group  $(\mathbb{Z}/p)^2$  embeds in  $M_{p^3}$  via  $(1, 0) \mapsto r^p$ ,  $(0, 1) \mapsto sr^p$  [EfMi11, Remark 8.1(c)]. Hence assumptions (I), (II) of §3 are satisfied.

For a profinite group  $\bar{G}$ , we consider  $\omega$  as a short exact sequence of trivial  $\bar{G}$ -modules. Then the Bockstein homomorphism  $\text{Bock} = \text{Bock}_{\bar{G}}: H^1(\bar{G}) \rightarrow H^2(\bar{G})$  is the connecting homomorphism in the associated long exact sequence of cohomology groups [NSW08, Th. 1.3.2].

**Proposition 11.1.** (a) For a continuous homomorphism  $\bar{\rho}: \bar{G} \rightarrow \mathbb{Z}/p$  we have  $\bar{\rho}^*(\alpha_\omega) = \text{Bock}(\bar{\rho})$ .

(b) For a continuous epimorphism  $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2): \bar{G} \rightarrow (\mathbb{Z}/p)^2$  we have  $\bar{\rho}^*(\alpha_{\omega'}) = \text{Bock}(\bar{\rho}_1) + \bar{\rho}_1 \cup \bar{\rho}_2$ .

(c) For a continuous homomorphism  $\bar{\rho}: \bar{G} \rightarrow (\mathbb{Z}/p)^2$  which is not surjective we have either  $\bar{\rho}^*(\alpha_{\omega'}) = \text{Bock}(\bar{\rho})$  (where  $\bar{\rho}$  is considered as an element of  $H^1(\bar{G})$ ), or  $\bar{\rho}^*(\alpha_{\omega'}) = 0$ .

*Proof.* (a) See [EfMi11, Prop. 9.2] for the case  $\bar{\rho} \neq 0$ . The case  $\bar{\rho} = 0$  is trivial.

(b) See [EfMi11, Prop. 9.4].

(c) We may assume that  $\bar{\rho} \neq 0$ , so  $\text{Im}(\bar{\rho}) \cong \mathbb{Z}/p$ . Then  $\bar{\rho}$  breaks via a commutative diagram

$$\begin{array}{ccccccc}
 & & & & \bar{G} & & \\
 & & & & \downarrow \bar{\rho} & & \\
 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & D & \longrightarrow & \mathbb{Z}/p \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & M_{p^3} & \longrightarrow & (\mathbb{Z}/p)^2 \longrightarrow 0,
 \end{array}$$

where  $D$  is either  $\mathbb{Z}/p^2$  or  $(\mathbb{Z}/p)^2$ . In the first case,  $\bar{\rho}^*(\alpha_{\omega'}) = \text{Bock}(\bar{\rho})$ . In the second case, the upper extension splits, so  $\bar{\rho}^*(\alpha_{\omega'}) = 0$ .  $\square$

Now let  $G$  be a profinite group and set  $\bar{G} = G/G^{(2,p)}$ .

**Lemma 11.2.** *Let  $\psi, \xi: G \rightarrow \mathbb{Z}/p$  be continuous homomorphisms such that  $\text{Bock}_G(\psi) = \psi \cup \xi$ . Let  $\bar{\psi}, \bar{\xi}: \bar{G} \rightarrow \mathbb{Z}/p$  be the homomorphisms induced by  $\psi, \xi$ , respectively. Then*

- (i)  $\bar{\psi}^*(\alpha_{\omega})$  is  $G$ -liftable; or
- (ii)  $\bar{\rho}^*(\alpha_{\omega'})$  is  $G$ -liftable, where  $\bar{\rho} = (\bar{\psi}, -\bar{\xi})$ .

*Proof.* (Compare [EfMi11, Prop. 10.2]). First assume that  $\text{Bock}_G(\psi) = 0$ . Since  $\bar{\psi}^*(\alpha_{\omega}) = \text{Bock}_{\bar{G}}(\bar{\psi})$  (Proposition 11.1(a)), the functoriality of  $\text{Bock}$  (as a connecting homomorphism) implies that  $\text{inf}_G(\bar{\psi}^*(\alpha_{\omega})) = \text{Bock}_G(\psi) = 0$ , i.e.,  $\bar{\psi}^*(\alpha_{\omega})$  is  $G$ -liftable, and (i) holds.

Next suppose that  $\text{Bock}_G(\psi) \neq 0$ . Then  $\psi \cup \xi \neq 0$ . As  $p > 2$ ,  $\psi \cup \psi = 0$ , so  $\psi, \xi$  are  $\mathbb{F}_p$ -linearly independent in  $H^1(G)$ . Hence  $\bar{\psi}, \bar{\xi}$  are  $\mathbb{F}_p$ -linearly independent in  $H^1(\bar{G})$ . Therefore  $\bar{\rho} = (\bar{\psi}, -\bar{\xi}): \bar{G} \rightarrow (\mathbb{Z}/p)^2$  is an epimorphism. One has  $\bar{\rho}^*(\alpha_{\omega'}) = \text{Bock}_{\bar{G}}(\bar{\psi}) - \bar{\psi} \cup \bar{\xi}$  (Proposition 11.1(b)), so by hypothesis,  $\text{inf}_G(\bar{\rho}^*(\alpha_{\omega'})) = 0$ , whence (ii).  $\square$

Let  $\pi: G \rightarrow \bar{G}$  be the natural projection. By Proposition 11.1,  $H^2(\bar{G})_{\pi}$  is the subgroup of  $H^2(\bar{G})$  generated by the elements of the forms  $\text{Bock}(\bar{\rho})$  and  $\text{Bock}(\bar{\rho}) + \bar{\rho} \cup \bar{\rho}'$  which vanish under inflation to  $H^2(G)$ .

The quotient  $\bar{G}$  is an elementary abelian  $p$ -group, so  $H^2(\bar{G})$  is generated by all Bockstein elements  $\text{Bock}(\bar{\psi})$  and all cup products  $\bar{\psi} \cup \bar{\psi}'$ , with  $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G})$  [EfMi11, Cor. 2.9(a)]. Therefore it is generated by the cohomology elements  $\text{Bock}(\bar{\rho})$  and  $\text{Bock}(\bar{\rho}) + \bar{\rho} \cup \bar{\rho}'$ .

We say that  $G$  satisfies the *Bockstein-cup kernel condition* if

$$\text{Ker}(H^2(\bar{G}) \xrightarrow{\text{inf}} H^2(G))$$

is generated by elements of the forms  $\text{Bock}(\bar{\rho})$  and  $\text{Bock}(\bar{\rho}) + \bar{\rho} \cup \bar{\rho}'$ .

Next we note that

$$\bar{T}(G) = T^{\mathbb{Z}/p}(G) = G^{(2,p)}, \quad T(G) = T^{\mathbb{Z}/p^2}(G) \cap T^{M_{p^3}}(G).$$

Since  $(\mathbb{Z}/p^2)^{(3,p)} = 1$  and  $(M_{p^3})^{(3,p)} = 1$  [EfMi11, Remark 8.1(b)], we deduce that

$$G^{(3,p)} \leq T(G) \leq G^{(2,p)}.$$

**Lemma 11.3.** *If  $S$  is a free profinite group, then  $S^{(3,p)} = T(S)$ .*

*Proof.* By Proposition 5.1,  $\langle \cdot, \cdot \rangle_{1, S^{(2,p)}}^A$  gives a perfect bilinear map as in the upper row of the following commutative diagram:

$$\begin{array}{ccc} S^{(2,p)}/S^{(3,p)} \times H^2(S/S^{(2,p)}) & \longrightarrow & \mathbb{Z}/p \\ \downarrow & & \parallel \\ S^{(2,p)}/T(S) \times H^2(S/S^{(2,p)})_{\pi} & \longrightarrow & \mathbb{Z}/p. \end{array}$$

The lower row of the diagram is perfect by Proposition 6.4(a). By Proposition 10.2, the right vertical map is an equality. Therefore the left vertical map is also an equality.  $\square$

We extend Lemma 11.3 as follows:

**Theorem 11.4.** *Let  $S$  be a free profinite group, let  $N$  be a closed normal subgroup of  $S$  contained in  $S^{(2,p)}$ , and let  $G = S/N$ . Then  $T(G) = G^{(3,p)}$  if and only if  $G$  satisfies the Bockstein-cup kernel condition.*

*Proof.* We have  $S/S^{(2,p)} \cong G/G^{(2,p)} = \bar{G}$  via  $\pi$ . Hence the Bockstein-cup kernel condition means that  $\text{Ker}(\text{inf}: H^2(S/S^{(2,p)}) \rightarrow H^2(S/N))$  is generated by elements of the forms  $\text{Bock}(\bar{\rho})$  and  $\text{Bock}(\bar{\rho}) + \bar{\rho} \cup \bar{\rho}'$ . In view of Remark 4.2(3), this is exactly the kernel generating condition for the subgroups  $N, S^{(2,p)}$  of  $S$ . By Theorem 7.5 (with  $G$  replaced by  $S$ ), it is equivalent to  $T(G) = \pi[T(S)]$ . In light of Lemma 11.3, the latter equality means that  $T(G) = \pi[S^{(3,p)}] = G^{(3,p)}$ .  $\square$

Let  $H^2(G)_{\text{cup, Bock}}$  be the subgroup of  $H^2(G)$  generated by cup products and Bockstein elements. Then Proposition 6.4 gives:

**Corollary 11.5.** *Let  $G$  be a profinite group satisfying the Bockstein-cup kernel condition. Then:*

(a) *There is a canonical perfect bilinear map*

$$G^{(2,p)}/G^{(3,p)} \times H^2(\bar{G})_{\text{cup, Bock}} \rightarrow \mathbb{Z}/p,$$

(b) *There is an exact sequence*

$$0 \rightarrow H^2(\bar{G})_{\text{cup, Bock}} \rightarrow H^2(\bar{G}) \rightarrow H^2(G/G^{(3,p)}).$$

*Proof.* Take a free profinite group  $S$  and a continuous epimorphism  $\pi: S \rightarrow G$  whose kernel  $N$  is contained in the Frattini subgroup of  $S$ . In particular,  $N \leq S^{(2,p)}$ , so by Theorem 11.4,  $T(G) = G^{(3,p)}$ . We now apply Proposition 6.4 with  $G$  replaced by  $S$ .  $\square$

## 12. APPLICATIONS TO ABSOLUTE GALOIS GROUPS

The next theorem assembles the facts in Galois cohomology which are needed for the intersection theorems for absolute Galois groups of [MSp96], [EfMi11], and [EfMi17], given in the Introduction. It is based on the injectivity part of the Merkurjev–Suslin theorem ([MS82], [GS06]), the identification of Bockstein elements as cup products [EfMi11, Prop. 2.6], as well as the structure of  $H^2$  for elementary abelian  $p$ -groups. While all ingredients of the proof essentially appear in the above-mentioned references, our new formalism demonstrates the common methodology which underlies all the previous proofs.

For a field  $F$  containing a root of unity  $\zeta$  of order  $p$ , whence the full group  $\mu_p$  of  $p$ th roots of unity, we identify  $\mu_p = \mathbb{Z}/p$  as  $G_F$ -modules, where  $\zeta$  corresponds to  $\bar{1} \in \mathbb{Z}/p$ . This gives the Kummer isomorphism  $F^\times/(F^\times)^p \xrightarrow{\sim} H^1(G_F)$ . Given  $a \in F^\times$ , let  $(a)_F \in H^1(G_F)$  correspond to the coset of  $a$  in  $F^\times/(F^\times)^p$ .

By [EfMi11, Prop. 2.6], for every  $\varphi \in H^1(G_F)$  one has

$$(12.1) \quad \text{Bock}_{G_F}(\varphi) = \varphi \cup (\zeta)_F.$$

**Theorem 12.1.** *Let  $F$  be a field containing a root of unity  $\zeta$  of order  $p$ , and let  $G = G_F$  be its absolute Galois group. Then:*

- (a)  *$G$  satisfies the 2-Massey kernel condition;*
- (b) *When  $p = 2$ ,  $G$  satisfies the 2-unitriangular kernel condition;*
- (c) *When  $p > 2$ ,  $G$  satisfies the Bockstein–cup kernel condition.*

*Proof.* We note that  $G_{(2,p)} = G^{(2,p)}$ . Set  $\bar{G} = G/G_{(2,p)} = G/G^{(2,p)}$ . Then  $H^1(\bar{G}) \cong H^1(G)$  via inflation. We recall that the 2-fold Massey product is just the usual cup product map  $\cup: H^1(\bar{G})^{\otimes 2} \rightarrow H^2(\bar{G})$  (Its image  $H^2(\bar{G})_{2\text{-Massey}}$  it is also denoted by  $H^2(\bar{G})_{\text{dec}}$ , for the *decomposable* part of  $H^2(G)$  [CEM12]).

(a) By the injectivity part of the Merkurjev–Suslin theorem, the kernel of the cup product map  $\cup: H^1(G)^{\otimes 2} \rightarrow H^2(G)$  is the “Steinberg group”,



generated by all tensor products  $(a)_F \otimes (1 - a)_F$ , with  $a \in F$ ,  $a \neq 0, 1$ . Therefore this kernel is also generated by all  $\varphi_1 \otimes \varphi_2$ , where  $\varphi_1, \varphi_2 \in H^1(G)$  satisfy  $\varphi_1 \cup \varphi_2 = 0$ . There is a commutative square

$$\begin{array}{ccc} H^1(\bar{G})^{\otimes 2} & \xrightarrow[\sim]{\text{inf}} & H^1(G)^{\otimes 2} \\ \cup \downarrow & & \downarrow \cup \\ H^2(\bar{G})_{2\text{-Massey}} & \xrightarrow{\text{inf}} & H^2(G). \end{array}$$

It follows that the kernel of the lower inflation map is generated by cup products from  $H^1(\bar{G})^{\otimes 2}$  which are in this kernel, as desired.

(b) We recall that for every prime  $p$ ,  $H^2(\bar{G})_{2\text{-ut}}$  is the subgroup of  $H^2(\bar{G})$  generated by all Bockstein elements and all cup products (see Proposition 11.1(a)). As  $p = 2$ ,  $\bar{G}$  is an elementary 2-abelian group, so  $H^2(\bar{G})$  is generated by cup products only [EfMi11, Cor. 2.9(b)]. Therefore  $H^2(\bar{G})_{2\text{-ut}} = H^2(\bar{G})_{2\text{-Massey}} = H^2(\bar{G})$ . The assertion therefore follows from (a).

(c) Consider  $\alpha$  in  $\text{Ker}(\text{inf}: H^2(\bar{G}) \rightarrow H^2(G))$ . It is a sum of Bockstein elements and cup products in  $H^2(\bar{G})$ . Therefore it can be written as

$$(12.2) \quad \alpha = \sum_i \bar{\psi}_i \cup \bar{\psi}'_i + (\text{Bock}_{\bar{G}}(\bar{\varphi}) - \bar{\varphi} \cup \overline{(\zeta)}_F),$$

with  $\bar{\psi}_i, \bar{\psi}'_i, \bar{\varphi} \in H^1(\bar{G})$ , and where  $\overline{(\zeta)}_F \in H^1(\bar{G})$  corresponds to the Kummer element  $(\zeta)_F \in H^1(G)$  under the inflation isomorphism  $H^1(\bar{G}) \cong H^1(G)$ . By (12.1),  $\text{Bock}_{\bar{G}}(\bar{\varphi}) - \bar{\varphi} \cup \overline{(\zeta)}_F$  is always in the above kernel. Hence  $\sum_i \bar{\psi}_i \cup \bar{\psi}'_i$  is also in the kernel. By (a), we may replace it by another such sum, to assume without loss of generality that each  $\bar{\psi}_i \cup \bar{\psi}'_i$  separately is in this kernel. Moreover,  $\text{Bock}_{\bar{G}}(\bar{\varphi}) - \bar{\varphi} \cup \overline{(\zeta)}_F = \text{Bock}_{\bar{G}}(\bar{\varphi}) + \bar{\varphi} \cup (-\overline{(\zeta)}_F)$ , is a pullback corresponding to the extension  $\omega'$  of §11 (Proposition 11.1(b)). Then in (12.2) all the summands are pullbacks in the kernel, as desired.  $\square$

We now recover in a uniform way the intersection theorems by Mináč, Spira, and the author ([EfMi17, Th. D], [MSp96, Cor. 2.18], [EfMi11]).

**Theorem 12.2.** *Let  $G = G_F$  be the absolute Galois group of a field  $F$  which contains a root of unity  $\zeta$  of order  $p$ . Then:*

- (a)  $G_{(3,p)}$  is the intersection of all closed normal subgroups  $M$  of  $G$  such that  $G/M$  is either  $\{1\}$ ,  $\mathbb{Z}/p$ , or  $\mathbb{U}_2(\mathbb{Z}/p)$ .
- (b) When  $p = 2$ ,  $G^{(3,2)}$  is the intersection of all closed normal subgroups  $M$  of  $G$  such that  $G/M$  is isomorphic to either  $\{1\}$ ,  $\mathbb{Z}/2$ ,  $\mathbb{Z}/4$ , or  $\mathbb{U}_2(\mathbb{Z}/2) = D_4$ .

- (c) When  $p > 2$ ,  $G^{(3,p)}$  is the intersection of all closed normal subgroups  $M$  of  $G$  such that  $G/M$  is isomorphic to either  $\{1\}$ ,  $\mathbb{Z}/p^2$ , or  $M_{p^3}$ .

Note that the quotient  $\{1\}$  is needed only when  $F$  is separably closed.

*Proof.* Take a presentation  $G = S/N$  of  $G$ , with  $S$  a free profinite group and  $N$  contained in its Frattini subgroup. Then  $N \leq S_{(2,p)} = S^{(2,p)}$ . Let  $\bar{G} = G/G_{(2,p)} = G/G^{(2,p)}$

- (a) Let  $\Omega$  consist of the single extension

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{U}_2(\mathbb{Z}/p) \rightarrow \bar{\mathbb{U}}_2(\mathbb{Z}/p) \rightarrow 1.$$

By Theorem 12.1(a) and Theorem 9.3(b),  $G_{(3,p)} = T^{\mathbb{U}_2(\mathbb{Z}/p)}(G)$ . The subgroups of  $\mathbb{U}_2(\mathbb{Z}/p)$  are  $\{1\}, \mathbb{Z}/p, (\mathbb{Z}/p)^2, \mathbb{U}_2(\mathbb{Z}/p)$ . Moreover, any closed normal subgroup of  $G$  with quotient  $(\mathbb{Z}/p)^2$  can be replaced in the intersection by two closed normal subgroups with quotient  $\mathbb{Z}/p$ .

- (b) Let  $\Omega$  consist of the central extensions

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0, \quad 0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{U}_2(\mathbb{Z}/2) = D_4 \rightarrow (\mathbb{Z}/2)^2 \rightarrow 1.$$

By Theorem 12.1(b) and Theorem 10.3(b),  $G^{(3,2)} = T(G) = T^{\mathbb{Z}/4}(G) \cap T^{D_4}(G)$ . The subgroups of  $\mathbb{Z}/4$  and  $D_4$  are  $\{1\}, \mathbb{Z}/2, \mathbb{Z}/4, (\mathbb{Z}/2)^2$ , and  $D_4$ . Again, a closed normal subgroup of  $G$  with quotient  $(\mathbb{Z}/2)^2$  can be replaced in the intersection by two closed normal subgroups with quotient  $\mathbb{Z}/2$ .

- (c) Let  $\Omega$  consist of the central sequences

$$\omega: 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0, \quad \omega': 0 \rightarrow \mathbb{Z}/p \rightarrow M_{p^3} \rightarrow (\mathbb{Z}/p)^2 \rightarrow 0.$$

By Theorem 12.1(c) and Theorem 11.4,  $G^{(3,p)} = T(G) = T^{\mathbb{Z}/p^2}(G) \cap T^{M_{p^3}}(G)$ . The subgroups of  $\mathbb{Z}/p^2$  and  $M_{p^3}$  are  $\{1\}, \mathbb{Z}/p, \mathbb{Z}/p^2, (\mathbb{Z}/p)^2$  and  $M_{p^3}$ . As before, the quotient  $(\mathbb{Z}/p)^2$  can be omitted from the list.

It remains to show that the quotient  $\mathbb{Z}/p$  can also be omitted from the list. To this end, let  $\psi: G \rightarrow \mathbb{Z}/p$  be a continuous epimorphism, and let  $\bar{\psi}: \bar{G} \rightarrow \mathbb{Z}/p$  be the induced epimorphism. By Lemma 11.2 and (12.1), one of the following holds:

Case (i):  $\bar{\psi}^*(\alpha_\omega)$  is  $G$ -liftable. Thus  $\psi$  lifts to a continuous homomorphism  $\rho: G \rightarrow \mathbb{Z}/p^2$ . The epimorphism in  $\omega$  is a Frattini cover [FJ08, Def. 22.5.1], so  $\rho$  is surjective. Consequently,  $\text{Ker}(\psi)$  contains a closed normal subgroup  $N$  of  $G$  with  $G/N \cong \mathbb{Z}/p^2$ .

Case (ii):  $(\bar{\psi}, -\overline{(\zeta)}_F)^*(\alpha_{\omega'})$  is  $G$ -liftable (where  $\overline{(\zeta)}_F \in H^1(\bar{G})$  is as in the previous proof). Thus  $(\psi, -(\zeta)_F)$  lifts to a continuous homomorphism  $\rho: G \rightarrow M_{p^3}$ . The epimorphism  $\omega'$  is also a Frattini cover, so  $\rho$  is surjective.

Then  $\psi$  is the composition of  $\rho$  with the epimorphism  $M_{p^3} \rightarrow \mathbb{Z}/p$ , given by  $r \mapsto 1$  and  $s \mapsto 0$ , so  $\text{Ker}(\psi)$  contains a closed normal subgroup  $N$  of  $G$  with  $G/N \cong M_{p^3}$ .

Consequently, in both cases, the normal subgroup  $\text{Ker}(\psi)$  of  $G$  with quotient  $\mathbb{Z}/p$  may be replaced by a normal closed subgroup  $N$  with  $G/N \cong \mathbb{Z}/p^2, M_{p^3}$ .  $\square$

The “*Kernel  $n$ -Unipotent Conjecture*” of Mináč and Tân [MT15] predicts that, when  $G = G_F$  is the absolute Galois group of a field  $F$  containing a root of unity of order  $p$ , the equality  $G_{(n+1,p)} = T^{\text{Un}(\mathbb{Z}/p)}(G)$  holds for every  $n \geq 2$ . By Remark 9.4 one can replace here  $G = G_F$  by its maximal pro- $p$  Galois group  $G_F(p)$ . This conjecture known in the following cases:

- (1)  $n = 2$  (Theorem 12.2(a)).
- (2) The maximal pro- $p$  quotient  $G(p)$  of  $G$  is a free pro- $p$  group ([Efr14a, Th. A], [MT15, Th. 2.7(a)], [Efr14b]).
- (3)  $F$  is  $p$ -rigid and  $p > 2$  [MT15, Th. 8.1].

For  $G = G_F$  as above, Theorem 9.3 cannot be applied to prove that  $G_{(n+1,p)} = T^{\text{Un}(\mathbb{Z}/p)}$  in any other situation. Indeed, the theorem assumes that the kernel  $N$  of the projection  $S \rightarrow G$  is contained in  $S_{(n,p)}$ , so unless (1) holds,  $N \leq S_{(3,p)}$ . However we have:

**Proposition 12.3.** *Let  $G = G_F$  be the absolute Galois group of a field  $F$  containing a root of unity of order  $p$ . Let  $S$  be a free profinite group, let  $N$  be a closed normal subgroup of  $S$  contained in  $S_{(3,p)}$ , and suppose that  $S/N \cong G$ . Then  $G(p)$  is a free pro- $p$  group.*

*Proof.* The projection  $S \rightarrow G$  induces an isomorphism  $S/S_{(3,p)} \cong G/G_{(3,p)}$ . By [EfMi17, Th. A] (which strengthens [CEM12]),  $H^2(G/G_{(3,p)})_{2\text{-Massey}} \cong H^2(G)$  via inflation. Since  $S$  is also realizable as an absolute Galois group [FJ08, Cor. 23.1.2] and  $H^2(S) = 0$ , we have in particular  $H^2(S/S_{(3,p)})_{2\text{-Massey}} = 0$ . Therefore  $H^2(G) = 0$ , whence  $H^2(G(p)) = 0$  [CEM12, Lemma 6.5]. Thus  $G(p)$  is a free pro- $p$  group [NSW08, Cor. 3.5.7].  $\square$

### 13. COUNTEREXAMPLES

The appendix of [MT15], by Mináč, Tân and the author, constructs a profinite group  $G$  such that  $G_{(3,p)} \neq T^{\text{Un}_2(\mathbb{Z}/p)}(G)$ . The following example, which uses the more recent results of [Efr17] and [Efr20a], extends this construction, and provides (under some assumptions on  $\Omega$ ) closed normal subgroups  $N$  of free profinite groups  $S$  such that  $N \leq \bar{T}(S)$  and  $T(G) \not\leq \pi[T(S)]$ , where  $\pi: S \rightarrow G = S/N$  is the natural projection. This shows

that the equivalent conditions of Theorem 7.5 need not hold in general. In the special case where  $\Omega = \{\omega_2\}$  (with notation as in §9) this recovers the above mentioned example from [MT15].

**Example 13.1.** Let  $S$  be the free profinite group on the basis  $X = \{x_1, \dots, x_k\}$  of  $k$  elements. We totally order  $X$  by setting  $x_1 < \dots < x_k$ . A word in the alphabet  $X$  is a *Lyndon word* if it is lexicographically smaller than all its proper suffixes. Fix  $n \geq 2$ . For a word  $w = (a_1 \cdots a_n)$  of length  $n$  in the alphabet  $X$ , consider the iterated commutator of length  $n$  in  $S$

$$\tau_w = [a_1, [a_2, \cdots [a_{n-2}, [a_{n-1}, a_n]] \cdots]].$$

Let  $\Omega$  be a set of central extensions as in §3 with  $m = p$  prime. Thus  $\bar{T}(S)/T(S)$  is an  $\mathbb{F}_p$ -linear space (Lemma 3.3). We assume:

- (1) The iterated commutators  $\tau_w$  for Lyndon words  $w$  of length  $n$  in  $X$  belong to  $\bar{T}(S)$ , and are  $\mathbb{F}_p$ -linearly independent in  $\bar{T}(S)/T(S)$ ;
- (2) One has  $k \geq |\mathbb{U}_\omega| + n - 1$  for every  $\omega \in \Omega$ .

For every  $n - 1 \leq i < j \leq k$ , the word  $w_{ij} = (x_1 x_2 \cdots x_{n-2} x_i x_j)$  is Lyndon. Set  $\tau_{ij} = \tau_{w_{ij}}$ . In view of (1), the cosets of the  $\tau_{ij}$  form a basis of a linear subspace  $W$  of  $\bar{T}(S)/T(S)$ . Let  $\varphi: W \rightarrow \mathbb{Z}/p$  be the linear map which maps each such coset to 1 (mod  $p$ ). Let  $N$  be the closed normal subgroup of  $S$  generated by  $\tau_{n-1,n} \tau_{ij}^{-1}$  for all  $n - 1 \leq i < j \leq k$ . Thus  $N \leq \bar{T}(S)$ , and  $NT(S)/T(S) \leq \text{Ker}(\varphi)$ , so  $\tau_{n-1,n} \notin NT(S)$ .

On the other hand, let  $\omega \in \Omega$  and let  $\rho: S/N \rightarrow \mathbb{U}_\omega$  be a continuous homomorphism. In light of (2), the pigeonhole principle yields  $n - 1 \leq i < j \leq k$  such that  $\rho(x_i N) = \rho(x_j N)$ . Then  $\rho(\tau_{n-1,n} N) = \rho(\tau_{ij} N) = 1$ . This shows that  $\tau_{n-1,n} N \in T(S/N)$ . Consequently,

$$T(S/N) \not\leq NT(S)/N,$$

i.e.,  $T(G) \not\leq \pi[T(S)]$  for the projection  $\pi: S \rightarrow G = S/N$ .

Specifically, in the  $p$ -Zassenhaus context, where  $\bar{T}(S) = S_{(n,p)}$  and  $T(S) = S_{(n+1,p)}$ , (1) holds by [Efr20b, Prop. 4.4]. In the lower  $p$ -central context, where  $\bar{T}(S) = S^{(n,p)}$  and  $T(S) = S^{(n+1,p)}$ , (1) holds by [Efr17, Th. 8.5(b)].

For (2) we simply take  $k$  sufficiently large.

Our second example refines [EfMi11, Example 13.5]:

**Example 13.2.** Theorem 12.2(b), and therefore also Theorem 12.1(b), do not extend to primes  $p > 2$ . Namely, there exist absolute Galois groups  $G = G_F$  as above such that  $G^{(3,p)} \neq \bigcap_{s=1}^2 T^{\mathbb{U}_s(\mathbb{Z}/p^{3-s})}(G)$ , i.e.,

$$(13.1) \quad G^{(3,p)} \neq T^{\mathbb{Z}/p^2}(G) \cap T^{\mathbb{U}_2(\mathbb{Z}/p)}(G).$$

We first claim that (13.1) holds whenever  $G$  is a profinite group such that  $G/G^{(3,p)}$  is non-abelian and is generated by two elements. Indeed, since  $\mathbb{U}_2(\mathbb{Z}/p)^{(3,p)} = 1$  and  $\mathbb{U}_2(\mathbb{Z}/p)$  is not generated by two elements, it is not a quotient of  $G$ . Further, all subgroups of  $\mathbb{Z}/p^2$  and  $\mathbb{U}_2(\mathbb{Z}/p)$ , except  $\mathbb{U}_2(\mathbb{Z}/p)$ , are abelian. Therefore  $G/(T^{\mathbb{Z}/p^2}(G) \cap T^{\mathbb{U}_2(\mathbb{Z}/p)}(G))$  is abelian, whence our claim.

Now let  $p > 2$ . [EfMi11, Example 13.5] gives a field  $F$  containing a root of unity of order  $p$ , such that  $G = G_F$  satisfies

$$G/G^{(3,p)} \cong \langle \mathbb{Z}/p^2 \rangle \rtimes \langle \mathbb{Z}/p^2 \rangle = \langle \tilde{\tau} \rangle \rtimes \langle \tilde{\sigma} \rangle,$$

where the generators  $\tilde{\tau}, \tilde{\sigma}$  satisfy  $\tilde{\sigma}\tilde{\tau}\tilde{\sigma}^{-1} = \tilde{\tau}^{1+p}$ . Therefore (13.1) holds.

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EARL KATZ FAMILY CHAIR IN PURE MATHEMATICS, DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, P.O. BOX 653, BE'ER-SHEVA 8410501, ISRAEL

*Email address:* efrat@bgu.ac.il