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SET FUNCTIONS ASSOCIATED WITH COMPOSITION OPERATORS AND CAPACITY INEQUALITIES

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Abstract. In this paper we give the detailed study of set functions associated with composition operator norms by using capacitary representations. On this way we prove that mappings generate bounded composition operators on Sobolev spaces are Lipschitz mappings in capacitory metrics of the hyperbolic type.

1. Introduction

Composition operators on Sobolev spaces arise in [20, 25] and represent the significant part of the geometric analysis of Sobolev spaces due its applications to the Sobolev embedding theory, that includes embedding theorems, extension theorems and traces theorems [4, 7, 8, 11, 15] and to the spectral theory of elliptic operators, see, for example, [10, 12]. In the composition operators theory the basic role represent set functions associated with the operator norms which was introduced in [27]. The result of [27] states: *Let $\varphi:\Omega\to\widetilde{\Omega}$ be a homeomorphism, where Ω and $\widetilde{\Omega}$ are domains in $\mathbb{R}^n$. Suppose there exists a collection pairwise disjoint open sets $\Omega_k\subset\Omega$, $\widetilde{\Omega}_k:=\varphi(\Omega_k)$, $k=1,2,...$, such that composition operators $\varphi^*_{\Omega_k}(f):=f\circ\varphi$,*

$$\varphi^*_{\Omega_k}:L^1_p(\widetilde{\Omega}_k)\to L^1_q(\Omega_k),\ 1\leq q<p\leq\infty,$$

are bounded. Then

$$\|\varphi^*_{\cup_{k=1}^{\infty}\Omega_k}\|^{\frac{1}{q}-\frac{1}{p}}=\sum_{k=1}^{\infty}\|\varphi^*_{\Omega_k}\|^{\frac{1}{q}-\frac{1}{p}}.$$

Hence [27], if $\varphi:\Omega\to\widetilde{\Omega}$ generates the bounded operator

$$\varphi^*:L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ 1\leq q<p\leq\infty,$$

by the composition rule $\varphi^*(f)=f\circ\varphi$, then the set function

$$\widetilde{\Phi}_{p,q}(\widetilde{A})=\sup_{f\in L^1_p(\widetilde{A})\cap C_0(\widetilde{A})}\left(\frac{\|\varphi^*(f)\mid L^1_q(\Omega)\|}{\|f\mid L^1_p(\widetilde{A})\|}\right)^{1/q-1/p},$$

is a bounded monotone countably additive set function (an outer measure) defined on open bounded subsets $\widetilde{A}\subset\widetilde{\Omega}$. The case $p=\infty$ was considered in [31].

In the present work we give characterizations of set functions in capacitary terms. This approach allows to define set functions as variation of capacity condensers. On this way the geometric theory of composition operators on Sobolev spaces [27, 32]

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is closely connected with various generalizations of quasiconformal mappings which are defined in capacity (moduli) terms and were studied by many authors, see, for example, [14, 16, 17, 19, 23, 24, 26].

Using capacitory representations of composition operators on Sobolev spaces we study capacitory metric distortions of weak (p,q)-quasiconformal mappings. Recall that a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ is called a weak (p,q)-quasiconformal mapping if $\varphi\in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion and

$$\int\limits_{\Omega}\left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}~dx<\infty~~\left(\operatorname*{ess\,sup}_{x\in\Omega}\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}<\infty,~\text{if}~p=q\right).$$

In the case $p=q=n$ we obtain usual quasiconformal mappings.

The quasiconformal mappings are bi-Lipschitz mappings in metrics associated with conformal capacity [1, 29]. In the present article we define a p-capacitary metric

$$d_p(x,y)=\inf_{\gamma}\operatorname{cap}_p^{\frac{1}{p}}(\gamma;\Omega),~~n-1<p\leq n,$$

where the infimum is taken over all continuous curves γ, joining points x and y in Ω. In the case $p=n$ metrics of such type was considered in [1] in a connection with quasiconformal mappings.

We prove that if $\varphi:\Omega\to\widetilde{\Omega}$ be a (p,q)-quasiconformal mapping, $n-1<q\leq p\leq n$, then, for any two points $x,y\in\widetilde{\Omega}$ the following inequality

$$d_q(\varphi^{-1}(x),\varphi^{-1}(y))\leq K_{p,q}(\varphi;\Omega)d_p(x,y),$$

holds. In the case $p=q=n$ we have the well known property of quasiconformal mappings [1, 3].

2. Composition operators on Sobolev spaces

2.1. **Sobolev spaces.** Let us recall the basic notions of the Sobolev spaces and the change of variable formula.

Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $W^1_p(\Omega)$, $1\leq p\leq\infty$, is defined [21] as a Banach space of locally integrable weakly differentiable functions $f:\Omega\to\mathbb{R}$ equipped with the following norm:

$$\|f\mid W^1_p(\Omega)\|=\|f\mid L_p(\Omega)\|+\|\nabla f\mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f, i. e. $\nabla f=(\frac{\partial f}{\partial x_1},...,\frac{\partial f}{\partial x_n})$.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1\leq p\leq\infty$, is defined as a space of locally integrable weakly differentiable functions $f:\Omega\to\mathbb{R}$ equipped with the following seminorm:

$$\|f\mid L^1_p(\Omega)\|=\|\nabla f\mid L_p(\Omega)\|.$$

In the Sobolev spaces theory, a crucial role is played by capacity as an outer measure associated with Sobolev spaces [21]. In accordance to this approach, elements of Sobolev spaces $W^1_p(\Omega)$ are equivalence classes up to a set of p-capacity zero [22].

The mapping $\varphi:\Omega\to\mathbb{R}^n$ belongs to the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega,\mathbb{R}^n)$, if its coordinate functions belongs to $W^1_{p,\mathrm{loc}}(\Omega)$. In this case, the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x\in\Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$, 2.2. **Composition operators.** Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^{*}: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ \ 1\leq q\leq p\leq\infty,$$

by the composition rule $\varphi^*(f)=f\circ\varphi$, if for any function $f\in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f)\in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\varphi;\Omega)<\infty$ such that

$$\|\varphi^*(f)\mid L^1_q(\Omega)\|\leq K_{p,q}(\varphi;\Omega)\|f\mid L^1_p(\widetilde{\Omega})\|.$$

Recall that the p-dilatation [2] of a Sobolev mapping $\varphi:\Omega\to\widetilde{\Omega}$ at a point $x\in\Omega$ is defined as

$$K_p(x)=\inf\{k(x):|D\varphi(x)|\leq k(x)|J(x,\varphi)|^{\frac{1}{p}}\}.$$

The following theorem gives the characterization of composition operators in terms of integral characteristics of mappings of finite distortion. Recall that a weakly differentiable mapping $\varphi:\Omega\to\mathbb{R}^n$ is the mapping of finite distortion if $D\varphi(x)=0$ for almost all x from $Z=\{x\in\Omega: J(x,\varphi)=0\}$ [29].

Theorem 2.1. *A homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^{*}: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ \ 1\leq q\leq p\leq\infty,$$

if and only if $\varphi\in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi;\Omega):=\|K_p\mid L_{\kappa}(\Omega)\|<\infty,\ 1/q-1/p=1/\kappa\ (\kappa=\infty,\ \textit{if}\ p=q).$$

The norm of the operator φ^* is estimated as $\|\varphi^*\|\leq K_{p,q}(\varphi;\Omega)$.

This theorem in the case $p=q$ was proved in [28] (see, also [5]), the case $q<p$ was proved in [27] and the limit case $p=\infty$ was considered in [9].

Let us recall, that homeomorphisms $\varphi:\Omega\to\widetilde{\Omega}$ which satisfy conditions of Theorem 2.1 are called as weak (p,q)-quasiconformal mappings [5, 30].

In the case of weak (p,q)-quasiconformal mappings, the following composition duality theorem holds [27] (the detailed proof can be found in [13]):

Theorem 2.2. *Let a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^{*}: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ \ n-1<q\leq p<\infty,$$

and has finite distortion in the case $p=\infty$, then the inverse mapping $\varphi^{-1}:\widetilde{\Omega}\to\Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^{*}: L^1_{q'}(\Omega)\to L^1_{p'}(\widetilde{\Omega}),$$

where $p'=p/(p-n+1)$, $q'=q/(q-n+1)$. In the case $n=2$ we have $1\leq q\leq p<\infty$, the inverse assertion is also correct and $p''=p$.

3. Capacitary description and set functions

3.1. **Set functions.** The geometric theory of composition operators on Sobolev spaces is based on the measure property of composition operators introduced in [27] (see, also [31]). This measure property use the additive set function that forms the basis of our results.

Let us define a set function $\widetilde{\Phi}_{p,q}(\widetilde{A})$, defined on open bounded subsets $\widetilde{A} \subset \widetilde{\Omega}$ and associated with a composition operator $\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega)$, $1 \leq q < p < \infty$,

$$\widetilde{\Phi}_{p,q}(\widetilde{A}) = \sup_{f \in L^1_p(\widetilde{A}) \cap C_0(\widetilde{A})} \left(\frac{\|\varphi^*(f) \mid L^1_q(\Omega)\|}{\|f \mid L^1_p(\widetilde{A})\|} \right)^{\kappa}, \quad 1/\kappa = 1/q - 1/p. \tag{3.1}$$

Theorem 3.1. [27] *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q < p \leq \infty.$$

Then function $\widetilde{\Phi}_{p,q}(\widetilde{A})$, defined by (3.1), is a bounded monotone countably additive set function defined on open bounded subsets $\widetilde{A} \subset \widetilde{\Omega}$.

In this section we study capacitary properties of the set functions associated with norms of composition operators. First of all, we recall the notion of a variational p-capacity [6, 29].

A condenser in a domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected disjoint closed relatively to Ω sets $F_0, F_1 \subset \Omega$. Then a continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1), if the set $F_i \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x | f(x) = i\}$, $i = 0, 1$.

The p-capacity of the condenser (F_0, F_1) relatively to domain Ω is the following quantity:

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \|f | L^1_p(\Omega)\|^p.$$

Here the greatest lower bound is taken over all functions admissible for the condenser $(F_0, F_1) \subset \Omega$. If the condenser has no admissible functions we put the capacity equal to infinity.

The next two theorems provide a capacitary description of composition operators.

Theorem 3.2. [5] *Let $1 < p < \infty$. A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_p(\Omega)$$

if and only if for every condenser $(F_0, F_1) \subset \widetilde{\Omega}$ the inequality

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq K_{p,p}(\varphi; \Omega) \operatorname{cap}_p^{1/p}(F_0, F_1; \widetilde{\Omega})$$

holds.

Theorem 3.3. [27] *Let $1 < q < p < \infty$. A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega)$$

if and only if there exists a bounded monotone countable-additive set function $\widetilde{\Phi}_{p,q}$ *defined on open subsets of* $\widetilde{\Omega}$ *such that for every condenser* $(F_0, F_1) \subset \widetilde{\Omega}$ *the inequality*

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq \widetilde{\Phi}_{p,q}(\widetilde{\Omega} \setminus F_0)^{\frac{p-q}{pq}} \operatorname{cap}_p^{1/p}(F_0, F_1; \widetilde{\Omega})$$

holds.

Further, one consider a distortion property of weak (p,q)-quasiconformal mappings for ring condensers. The ring condenser R (see, for example, [2, 18]) in a domain $\Omega \subset \mathbb{R}^n$ is the pair $R = (F; G)$ of sets $F \subset G \subset \Omega$, where F is a connected compact set and G is a connected open set. The p-capacity of the ring $R = (F; G)$ is defined by

$$\operatorname{cap}_p(F; G) = \inf \|f | L^1_p(\Omega)\|^p,$$

where the greatest lower bound is taken over all continuous functions $f \in L^1_p(\Omega)$ such that $f \geq 1$ on F and $f = 0$ on $\Omega \setminus G$. Next we show that in the case $n-1 < q \leq p < \infty$ we can consider ring condensers only.

3.2. **Capacitory characterizations of set functions.** To give characterizations of the set function $\widetilde{\Phi}_{p,q}$ in capacitary terms, we have to define another set function $\widetilde{\Psi}$ and consider its variation. Let $1 < q < p < \infty$ and $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism. We define the set function on open bounded subsets $\widetilde{A} \subset \widetilde{\Omega}$ by the rule

$$\widetilde{\Psi}_{p,q}(\widetilde{A}) = \sup_{(F;G) \subset \widetilde{A}} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F); \varphi^{-1}(G))}{\operatorname{cap}_p^{\frac{1}{p}}(F; G)} \right)^{\frac{pq}{p-q}},$$

where the supremum is taken over all ring condensers $(F; G) \subset \widetilde{A}$ such that $\operatorname{cap}_p(F; G) \neq 0$.

For function $\widetilde{\Psi}_{p,q}$, we consider its variation. Variation of a set function is a quasiadditive set function

$$V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) = \sup_{\{\widetilde{A}_k\}} \sum_{k=1}^{\infty} \widetilde{\Psi}_{p,q}(\widetilde{A}_k),$$

where the supremum is taken over all families of disjoint sets $\{\widetilde{A}_k\}_{k \in \mathbb{N}}$, $\widetilde{A}_k \subset \widetilde{\Omega}$.

Proposition 3.4. *Let* $n-1 < q < p < \infty$*, then the set function* $\widetilde{\Psi}_{p,q}$ *is positive and non-decreasing.*

Proof. Let $\widetilde{A}_1 \subset \widetilde{A}_2 \subset \widetilde{\Omega}$. Then

$$\begin{aligned}
\widetilde{\Psi}_{p,q}(\widetilde{A}_1) = & \sup_{(F_1;G_1) \subset \widetilde{A}_1} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_1); \varphi^{-1}(G_1))}{\operatorname{cap}_p^{\frac{1}{p}}(F_1; G_1)} \right)^{\frac{pq}{p-q}} \\
& = \sup_{(F_1;G_1) \subset \widetilde{A}_2} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_1); \varphi^{-1}(G_1))}{\operatorname{cap}_p^{\frac{1}{p}}(F_1; G_1)} \right)^{\frac{pq}{p-q}} \\
& \leq \sup_{(F_2;G_2) \subset \widetilde{A}_2} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_2); \varphi^{-1}(G_2))}{\operatorname{cap}_p^{\frac{1}{p}}(F_2; G_2)} \right)^{\frac{pq}{p-q}} = \widetilde{\Psi}_{p,q}(\widetilde{A}_2),
\end{aligned}$$

because any condenser $(F_1; G_1) \subset \widetilde{A}_2$ coincides with some condenser $(F_2; G_2) \subset \widetilde{A}_2$.

Let us proof, that for arbitrary open set $\widetilde{A} \subset \mathbb{R}^n$ the function $\widetilde{\Psi}_{p,q}(\widetilde{A})$ is positive. Fix a point $x \in \widetilde{A}$ and choose $r > 0$ such that ring condenser $(\overline{B}(x,r); B(x,2r))$ is a subset of $\widetilde{A}$. Then, by [6],

$$\operatorname{cap}_p(\overline{B}(x,r); B(x,2r)) \geq c_1 > 0.$$

The preimage of $\overline{B}(x,r)$ is a continuum set. Therefore, because in the case $q > n-1$ the capacity of a continuum set is positive [6], we have

$$\operatorname{cap}_p(\varphi^{-1}(\overline{B}(x,r)); \varphi^{-1}(B(x,2r))) \geq c_2 > 0.$$

Hence, the set function $\widetilde{\Psi}_{p,q}(\widetilde{A}) > 0$ for an arbitrary open set $\widetilde{A} \subset \mathbb{R}^n$.

□

Remark 3.5. *In the case $n = 2$ Proposition 3.4 is correct for all $1 \leq q < p < \infty$.*

The following statement establishes a correspondence between set functions $\widetilde{\Phi}_{p,q}$ and $\widetilde{\Psi}_{p,q}$.

Theorem 3.6. *Fix a constant $M < \infty$ and let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism. Then the following conditions are equivalent:*

(1) *there exists a bounded monotone countable-quasiadditive set function $\widetilde{\Phi}_{p,q}$ defined on open subsets of $\widetilde{\Omega}$ such that for every ring condenser $(F; \widetilde{A}) \subset \widetilde{\Omega}$ the inequality*

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(F); \varphi^{-1}(\widetilde{A})) \leq \widetilde{\Phi}_{p,q}(\widetilde{A})^{\frac{p-q}{pq}} \operatorname{cap}_p^{1/p}(F; \widetilde{A})$$

holds and $\widetilde{\Phi}_{p,q}(\widetilde{\Omega}) \leq M$;

(2) *function $\widetilde{\Psi}_{p,q}$ has a bounded variation $V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) \leq M$;*

(3) *for every family of disjoint ring condensers $\{(F_k; \widetilde{A}_k)\}_{k\in\mathbb{N}}$ in $\widetilde{\Omega}$ with $\operatorname{cap}_p^{\frac{1}{p}}(F_k; \widetilde{A}_k) \neq 0$, the following estimate is true*

$$\sum_{k=1}^{\infty} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_k); \varphi^{-1}(\widetilde{A}_k))}{\operatorname{cap}_p^{\frac{1}{p}}(F_k; \widetilde{A}_k)} \right)^{\frac{pq}{p-q}} \leq M. \tag{3.2}$$

Proof. First we show, that the second and the third condition are equivalent. If $V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) \leq M$ then, by virtue of definitions of $\widetilde{\Psi}$ and $V(\widetilde{\Psi}_{p,q})$, we have

$$\sup_{\{\widetilde{A}_k\}} \sum_{k=1}^{\infty} \sup_{(F_k; G_k) \subset \widetilde{A}_k} \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_k); \varphi^{-1}(G_k))}{\operatorname{cap}_p^{\frac{1}{p}}(F_k; G_k)} \right)^{\frac{pq}{p-q}} \leq M,$$

where the first supremum is taken over all families of disjoint sets $\widetilde{A}_k$, and the second is taken over all ring condensers in $\widetilde{A}_k$ with non-zero capacity. Hence, for an arbitrary family $\{\widetilde{A}_k\}_{k\in\mathbb{N}}$ and condensers $(F_k; \widetilde{A}_k)$, the sum (3.2) is bounded by the same constant M.

Now we assume that condition 3 is fulfill. For an arbitrary family of disjoint sets $\{(F_k; \widetilde{A}_k)\}_{k\in\mathbb{N}}$ in $\widetilde{\Omega}$ and an arbitrary $\varepsilon > 0$, we choose condensers $(F_k; G_k)$ such that

$$\widetilde{\Psi}_{p,q}(\widetilde{A}_k) < \left(\frac{\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_k); \varphi^{-1}(G_k))}{\operatorname{cap}_p^{\frac{1}{p}}(F_k; G_k)} \right)^{\frac{pq}{p-q}} + \frac{\varepsilon}{2^k}.$$

Then $\sum\limits_{k=1}^{\infty} \widetilde{\Psi}_{p,q}(\widetilde{A}_k) < M + \varepsilon$ and, therefore, $V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) \leq M$ and condition 2 is fulfill.

Next we show that condition 1 and 2 are equivalent. If condition 1 is true, then, due to the capacitary inequality for $\widetilde{\Phi}_{p,q}$, $\widetilde{\Psi}_{p,q}(\widetilde{A}) \leq \sup \widetilde{\Phi}_{p,q}(G)$, where the supremum is taken over all $G \subset \widetilde{A}$. Hence, by monotonicity of $\widetilde{\Phi}_{p,q}$, $\widetilde{\Psi}_{p,q}(\widetilde{A}) \leq \widetilde{\Phi}_{p,q}(\widetilde{A})$ for every $\widetilde{A} \subset \widetilde{\Omega}$. Further, by the definition of the variation,

$$V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) = \sup_{\{\widetilde{A}_k\}} \sum_{k=1}^{\infty} \widetilde{\Psi}_{p,q}(\widetilde{A}_k) \leq \sup_{\{\widetilde{A}_k\}} \sum_{k=1}^{\infty} \widetilde{\Phi}_{p,q}(\widetilde{A}_k).$$

Function $\widetilde{\Phi}_{p,q}$ is countable-quasiadditive and monotone, which implies

$$V(\widetilde{\Psi}_{p,q}, \widetilde{\Omega}) \leq \sup \widetilde{\Phi}_{p,q}(\bigcup_{k=1}^{\infty} \widetilde{A}_k) \leq \widetilde{\Phi}_{p,q}(\widetilde{\Omega})$$

and condition 2 holds.

If condition 2 holds, then for an arbitrary condenser $(F, \widetilde{A})$ with $\operatorname{cap}_p(F, \widetilde{A}) \neq 0$ by the definition of $\widetilde{\Psi}_{p,q}$ and $V(\widetilde{\Psi}_{p,q})$ we obtain the inequality

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(F); \varphi^{-1}(\widetilde{A})) \leq V(\widetilde{\Psi}_{p,q}, \widetilde{A}) \operatorname{cap}_p^{1/p}(F; \widetilde{A}).$$

Function $V(\widetilde{\Psi}_{p,q})$ is quasiadditive and for finishing the proof we have to show that the above inequality holds also for the case $\operatorname{cap}_p(F; \widetilde{A}) = 0$. For an arbitrary condenser $(F; \widetilde{A})$ with $\operatorname{cap}_p(F; \widetilde{A}) = 0$, let us consider a decreasing sequence of compact sets $\{F_k\}_{k=1}^{\infty}$, $F_k \subset \widetilde{A}$, $F = \bigcap_{k=1}^{\infty} F_k$, with $\operatorname{cap}_p(F_k; \widetilde{A}) > 0$ (for example, $F_k = \{x : \operatorname{dist}(x, F) \leq 1/k\}$). For such condensers

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(F_k); \varphi^{-1}(\widetilde{A})) \leq V(\widetilde{\Psi}_{p,q}, \widetilde{A}) \operatorname{cap}_p^{1/p}(F_k; \widetilde{A}).$$

Due to the continuity of the capacity we can pass to the limit and obtain the above inequality for condensers with zero capacity. □

Note, that in [16] the similar statement was proven in the particular case $p = n$.

4. Capacitary metrics

In this subsection we introduce capacitary metric of the hyperbolic type and prove, that (p, q)-quasiconformal mappings are Lipschitz mappings in correspondence capacitary metrics.

Let $n-1 < p \leq n$ ($1 \leq p \leq 2$ if $n = 2$) and let a domain $\Omega \subset \mathbb{R}^n$ have a boundary of non-zero p-capacity.

Let us define a capacitary metric

$$d_p(x, y) = \inf_{\gamma} \operatorname{cap}_p^{\frac{1}{p}}(\gamma; \Omega),$$

where the infimum is taken over all continuous curves γ, joining points x and y in Ω. In the case $p = n$ metrics of such type was considered in [1] in a connection with quasiconformal mappings.

Theorem 4.1. *Let $n - 1 < p \leq n$ ($1 \leq p \leq 2$ if $n = 2$) and $\operatorname{cap}_p(\partial\Omega) > 0$ then*

$$d_p(x, y) = \inf_{\gamma} \operatorname{cap}_p^{\frac{1}{p}}(\gamma; \Omega)$$

is a metric.

Proof. The restriction on p arises because for $p > n$ only empty set has zero capacity and for $p \leq n-1$ even a continuum has zero capacity. If boundary $\partial\Omega$ has zero capacity, then, by the definition of $\operatorname{cap}_p^{\frac{1}{p}}(\gamma;\Omega)$ will be equal to zero for any continuous curve γ.
1. **Identity**:
If $x = y$ then the infimum is reached on one-point set, which capacity is zero, hence, $d_p(x,y) = 0$. Now, let $d_p(x,y) = 0$. Suppose that $|x-y| > 0$. Then by the capacity estimate [16] for any continuous curve $\gamma \subset \Omega$ which joint points x and y we have

$$\operatorname{cap}_p^{n-1}(\gamma;\Omega) \geq c(n,p)\frac{|x-y|^p}{|\Omega|^{p-n+1}} > 0.$$

Contradiction.
2. **Summetry**: Follows from the definition $d_p(x,y)$:

$$d_p(x,y) = \inf_{\gamma} \operatorname{cap}_p^{\frac{1}{p}}(\gamma;\Omega) = d_p(y,x)$$

2. **Triangle inequality**: Follows from the triangle inequality for L^1_p-seminorms in the definition of the capacity. □

By using Theorem 3.3 we obtain the following statement.

Theorem 4.2. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a (p,q)-quasiconformal mapping, $n-1 < q \leq p \leq n$ ($1 \leq q \leq p \leq 2$ if $n = 2$). Then, for any two points $x, y \in \widetilde{\Omega}$ the following inequality*

$$d_q(\varphi^{-1}(x), \varphi^{-1}(y)) \leq K_{p,q}(\varphi;\Omega) d_p(x,y),$$

holds.

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a (p,q)-quasiconformal mapping, then there exists a bounded monotone countable-additive set function $\widetilde{\Phi}_{p,q}$ defined on open subsets of $\widetilde{\Omega}$ such that for every condenser $(F_0, F_1) \subset \widetilde{\Omega}$ the inequality

$$\begin{aligned}\operatorname{cap}_q^{1/q}(\varphi^{-1}(F_0), \varphi^{-1}(F_1);\Omega) &\leq \widetilde{\Phi}_{p,q}(\widetilde{\Omega} \setminus F_0)^{\frac{p-q}{pq}} \operatorname{cap}_p^{1/p}(F_0, F_1;\widetilde{\Omega}) \\ &\leq K_{p,q}(\varphi;\Omega) \operatorname{cap}_p^{1/p}(F_0, F_1;\widetilde{\Omega})\end{aligned}$$

holds.

Let $\widetilde{\gamma} \subset \widetilde{\Omega}$ be a continuous curve which join x and y in $\widetilde{\Omega}$. Because φ is a homeomorphism, then $\varphi^{-1}(\widetilde{\gamma})$ is a continuous curve which join $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ in Ω. Hence we obtain the following inequalities:

$$d_q(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(\widetilde{\gamma});\Omega) \leq K_{p,q}(\varphi;\Omega) \operatorname{cap}_p^{\frac{1}{p}}(\widetilde{\gamma};\widetilde{\Omega}).$$

Now since $\widetilde{\gamma}$ is an arbitrary continuous curve which join x and y, then, taking the infimum over all such $\widetilde{\gamma}$, we obtain

$$d_q(\varphi^{-1}(x), \varphi^{-1}(y)) \leq K_{p,q}(\varphi;\Omega) \inf \operatorname{cap}_p^{\frac{1}{p}}(\widetilde{\gamma};\widetilde{\Omega}) = K_{p,q}(\varphi;\Omega)\ d_p(x,y).$$

□

Using the composition duality theorem (Theorem 2.2) for composition operators, we immediately obtain the following result:

Theorem 4.3. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a (p,q)-quasiconformal mapping, $n < q \leq p < \infty$. Then, for any two points $x, y \in \Omega$ the following inequality*

$$d_{p'}(\varphi(x), \varphi(y)) \leq K_{q',p'}(\varphi^{-1}; \widetilde{\Omega})\ d_{q'}(x,y),\ p' = \frac{p}{p-n+1}, q' = \frac{q}{q-n+1}$$

holds.

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